CS455 HW3

Due Thursday, 24 February 2005

Question 1

The standard way to compute a rotation about an arbitrary axis through the origin is to concatenate five rotation matrices. This note shows how vector algebra makes it easy to rotate about an arbitrary axis in a single step.

Figure 1 shows a point **P** which we want to rotate an angle θ about an axis that passes through **B** with a direction defined by unit vector **n**. So, given the angle θ, the unit vector **n**, and Cartesian coordinates for the points **P**, **B**, we want to find Cartesian coordinates for the point $\mathbf{P}'$.

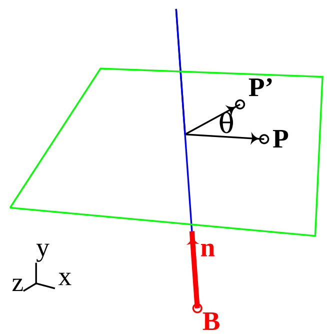


Figure 1: Rotation about an Arbitrary Axis

The key insight needed is shown in Figure 2. Let **u** and **v** be any two three-dimensional

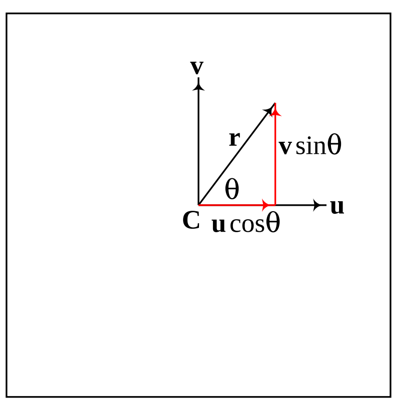


Figure 2: Key Insight

vectors that satisfy $u \cdot v = 0$ (that is, they are perpendicular) and $|\mathbf{u}| = |\mathbf{v}| \neq 0$ (that is, they are they same length but not necessarily unit vectors). We want to find a vector **r** that is obtained by rotating **u** an angle θ in the plane defined by **u** and **v**. As suggested in

Figure 2,

$$\mathbf{r} = \mathbf{u}\cos\theta + \mathbf{v}\sin\theta. \tag{1}$$

With that insight, it is easy to compute a rotation about an arbitrary axis. Referring

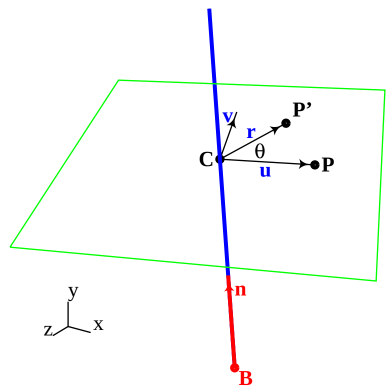


Figure 3: Rotation about an Arbitrary Axis

to Figure 3, we compute

$$\mathbf{C} = \mathbf{B} + [(\mathbf{P} - \mathbf{B}) \cdot \mathbf{n}]\mathbf{n}. \tag{2}$$

$$\mathbf{u} = \mathbf{P} - \mathbf{C} \tag{3}$$

$$\mathbf{v} = \mathbf{n} \times \mathbf{u} \tag{4}$$

Then, $\mathbf{r}$ is computed using equation (1), and

$$\mathbf{P}' = \mathbf{C} + \mathbf{r}. \tag{5}$$

It is possible to take these simple vector equations and to create from them a single 4×4 transformation matrix for rotation about an arbitrary axis. Let $\mathbf{P} = (x, y, z)$, $\mathbf{P}' = (x', y', z')$, $\mathbf{B} = (B_x, B_y, B_z)$, and $\mathbf{n} = (n_x.n_y, n_z)$. We seek a 4×4 matrix M such that

$$M \left\{ \begin{array}{c} x \\ y \\ z \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} x' \\ y' \\ z' \\ 1 \end{array} \right\}$$

Find the elements of M in terms of $\sin\theta$, $\cos\theta$, $B_x, B_y, B_z, n_x.n_y$, and n_z.

Question 2:

A sphere with center $(5, 3, 7)$ and radius 13 is intersected with a ray defined by $\mathbf{R}(t) = (x, y, z) = (16, 11, 35) + (-2, -1, -4)t$. (This vector is V in the above formulation). A point light source is located at $(30, 0, 0)$. The ray intersection is being performed for the purpose of ray casting: the eye location is $(16, 11, 35)$ and the vector $(-2, -1, -4)$ is chosen so as to compute the color of a certain pixel.

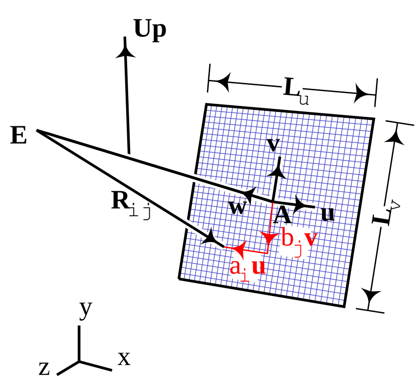
2.1. What are the values of t at which the ray intersects the sphere?

2.2. What are the values of $\mathbf{H} = (x, y, z)$ of the nearest point at which the ray intersects the sphere?
2.3. What is the unit normal vector at that intersection point?

2.4. What is the reflection vector R at which the ray bounces off of the sphere?
2.5. What is the cosine of the angle between the normal vector $\mathbf{N}$ and the vector $\mathbf{L} - \mathbf{H}$?

Question 3:
A triangle has vertices $\mathbf{P}_s = (1, 2, 3)$, $\mathbf{P}_t = (3, -2, 4)$, $\mathbf{P}_u = (5, 3, -1)$. The Barycentric coordinates (s, t, u) of a point P lying in the plane of the triangle are defined such that $s + t + u = 1$ and $P = s\mathbf{P}_s + t\mathbf{P}_t + u\mathbf{P}_u$.
3.1. Find the intersection between the triangle and the ray $(18, 21, 27) + (-4, -5, -6)r$ (we are using r for the ray parameter instead of t because t is being used as a Barycentric coordinate). In your answer, give r, (x, y, z), and (s, t, u).
3.2. If the normal vectors at $\mathbf{P}_s$, $\mathbf{P}_t$, and $\mathbf{P}_u$ are $(3, 4, 12)/13$, $(-3, 4, 12)/13$ and $(3, -4, 12)/13$ respectively, what is the unit normal vector at the intersection point (approximated using the Barycentric coordinates)?
3.3. If the vector to the light is $(1, 2, 3)$ and the color of the triangle is (r,g,b)=(0,1,1), what color would the pixel corresponding to this ray be painted, using the simplified Phong lighing model with n=5?

Question 4:
An eye position is given as $E = (16, 11, 35)$ in world coordinates. The Look-at position is given as $A = (3, 4, 5)$ in world coordinates. The Up vector is $(0, 1, 0)$ in world coordinates. The number of pixels that make up the ray-traced image is 500 rows and 500 columns. The tangent of the view half-angle is 0.25 (the half-angle is the angle between a ray from E to A and a ray from E to the edge of the projection plane.)



4.1. What are the u, v, and w unit vectors in this Eye coordinate system?
4.2. What are the (u,v,w) coordinates of the eye?
4.3. What are the (u,v,w) coordinates of the look-at position?
4.4. Give an equation for the ray that starts at E and passes through the center of pixel (i,j).