

Best linear common divisors for approximate degree reduction

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Two or more randomly chosen polynomials rarely have a common factor. The paper examines how a given set of polynomials can be perturbed so as to induce a linear common factor. An algorithm is derived for determining such a set of perturbation polynomials which have the guaranteed minimum norm for all possible such perturbations over a prescribed parameter interval. This result is applied to the problem of approximately reducing the degree of a rational curve.

degree reduction, approximation, polynomial GCD

A polynomial $f(t)$ with real coefficients is said to have a root $t = \tau$ if $f(\tau) = 0$. This is equivalent to saying that $(t - \tau)$ is a linear factor or a linear divisor of $f(t)$. A set of polynomials $\{f_1(t), \dots, f_m(t)\}$, $m \geq 2$, with coefficients chosen at random are unlikely to have a linear common factor $(t - \tau) = 0$. That is, there is unlikely to exist a number τ for which $f_1(\tau) = f_2(\tau) = \dots = f_m(\tau) = 0$. Further, if those coefficients are defined as floating-point numbers, it can be argued that the classical notion of common factors is meaningless, because of the uncertainty inherent in floating-point numbers¹.

In many practical problems, the concept of a precise common factor can be replaced to good advantage with the notion of a nearest common divisor^{2,3}. We state our problem formally as follows.

Problem: Given a set of m polynomials

$$f(t) = \{f_1(t), \dots, f_m(t)\} \quad m \geq 2 \quad (1)$$

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with $\|f_i(t)\|_{[a,b]} = \max_{a \leq t \leq b} (|f_i(t)|) = 1$, $i = 1, 2, \dots, m$, find a set of perturbation polynomials

$$e(t) = \{e_1(t), \dots, e_m(t)\} \quad (2)$$

such that the polynomials $\{f_1(t) + e_1(t), \dots, f_m(t) + e_m(t)\}$ have a linear common factor, and

$$\|e(t)\|_{[a,b]} = \max_{a \leq t \leq b} \left(\sum_{i=1}^m e_i^2(t) \right)^{1/2} \quad (3)$$

is the minimum for all possible perturbations.

There are two impelling motivations for this nearest-common-divisor problem. First, floating-point arithmetic is widely used in scientific computation, and, whereas the notion of an exact common divisor cannot be sustained in a floating-point environment, nearest common divisors lend themselves well to floating-point arithmetic. Second, many, if not most, practical problems in science and engineering are quite satisfied with approximate solutions that are correct within some tolerance. In some problems in which the question of common divisors arises, an exact arithmetic implementation of Euclid's algorithm rarely detects any common factor, whereas a nearest common divisor often exists which requires only minor perturbations over the $[a, b]$ interval of interest. For example, the cubic polynomials

$$f_1(t) = (t-6)(t+3)(t+7) = t^3 + 4t^2 - 39t - 126 \quad (4)$$

$$f_2(t) = (t-15)(t+5)(t+10) = t^3 - 175t - 750 \quad (5)$$

in Figure 1 have roots of $\{-7, -3, 6\}$ and $\{-10, -5, 15\}$, respectively, with nothing that is even close to a common root. Further, their resultant (which equals zero if and only if they have a common root) turns out to be 52690176, which one would not interpret as signaling the proximity of a common root. However, if we perturb f_1 and f_2 by cubic polynomials e_1 and e_2 , respectively,

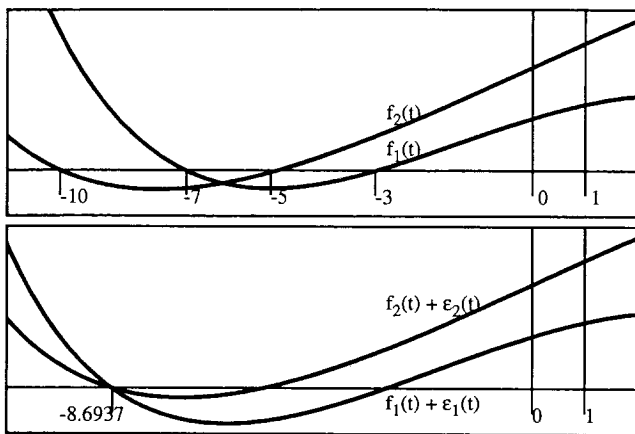


Figure 1. Example

where, over the $[0, 1]$ interval, $\|\varepsilon_1\|/\|f_1\| = 0.000036$ and $\|\varepsilon_2\|/\|f_2\| = 0.000049$, a common root is coerced to occur at $t = -8.6937$, as shown. Thus, the functional value of f_1 and f_2 is altered by less than 0.0004%.

We show that the nearest-common-divisor problem has an analytic solution. That is, we can write a precise description of the perturbation polynomials which have the smallest possible norms. For clarity, we derive the solution for the domain interval $[-1, 1]$, which can easily be mapped to an arbitrary interval $[a, b]$. The solution proceeds by determining what the minimum perturbation is to force a set of polynomials $f(t)$ to have a common root at a specified value of t (that is discussed in the second section of the paper). The third section shows how to determine which of all the real values of t requires the smallest possible perturbation (as defined in Equation 3) to become the common root.

One application of the best linear common divisor solution is the problem from computer-aided geometric design of approximately reducing the degree of a rational curve, an example of which is provided in the fourth section of the paper.

PIPE PROBLEM

We begin by examining what we refer to as the univariate ‘pipe’ problem: find the polynomial $\varepsilon(t)$ of degree of at most n which interpolates a single point (α, γ) , and which has the smallest uniform norm $\|\varepsilon(t)\|_{[-1, 1]}$. We refer to this as the pipe problem, because it can be regarded as the problem of finding the degree $\leq n$ polynomial passing through (α, γ) whose graph can be contained in the narrowest horizontal ‘pipe’ centered on the x axis and extending $-1 \leq t \leq 1$ as shown in Figure 2.

If $\alpha \in [-1, 1]$, a nonunique solution is provided by any polynomial whose largest value in $[-1, 1]$ occurs at (α, γ) . The simplest such solution is obviously the horizontal line $\varepsilon(t) = \gamma$. For this, and any other solution for the case $\alpha \in [-1, 1]$, $\|\varepsilon\| = |\gamma|$. If $\alpha \notin [-1, 1]$,

$$\varepsilon(t) = \frac{\gamma}{T_n(\alpha)} T_n(t) \tag{6}$$

and

$$\|\varepsilon(t)\| = \left| \frac{\gamma}{T_n(\alpha)} \right| < |\gamma| \tag{7}$$

In words, $\varepsilon(t)$ is the n th Chebyshev polynomial, scaled to interpolate (α, γ) .

We prove this through contradiction by supposing that there exists a solution $\hat{\varepsilon}(t)$ to the pipe problem for which $\|\hat{\varepsilon}\| < \|\varepsilon\|$. Consider the difference

$$d(t) = \varepsilon(t) - \hat{\varepsilon}(t) \tag{8}$$

Recall that the Chebyshev polynomial of degree n is defined by

$$T_n(t) = \cos(n \arccos t) \tag{9}$$

which can be defined by the recurrence

$$\begin{aligned} T_0(t) &= 1 \\ T_1(t) &= t \\ T_{n+1}(t) &= 2tT_n(t) - T_{n-1}(t) \quad n = 1, 2, 3, \dots \end{aligned} \tag{10}$$

For example, this recurrence leads to

$$\begin{aligned} T_2(t) &= 2t^2 - 1 \\ T_3(t) &= 4t^3 - 3t \\ T_4(t) &= 8t^4 - 8t^2 + 1 \\ T_5(t) &= 16t^5 - 20t^3 + 5t \end{aligned} \tag{11}$$

As shown in Figure 3 and discussed in Reference 4, p 61, $\varepsilon(t)$ has n roots in the interval $[-1, 1]$ at values $t_k = \cos(((2k - 1)/2n)\pi)$, $k = 1, 2, \dots, n$. Also, $\varepsilon(t)$ has $n + 1$ extrema in the interval $[-1, 1]$ at points

$$t'_k = \cos \frac{k}{n} \pi \quad k = 0, 1, \dots, n \tag{12}$$

where

$$\varepsilon(t'_k) = (-1)^k \|\varepsilon\| \tag{13}$$

Since we assume that $\|\hat{\varepsilon}\|_{[-1, 1]} < \|\varepsilon\|_{[-1, 1]}$, it is clear that $\hat{\varepsilon}$ cannot touch the pipe. Thus, the signs of the $n + 1$ values $d(t'_k)$ must alternate. This implies that $d(t)$ has at

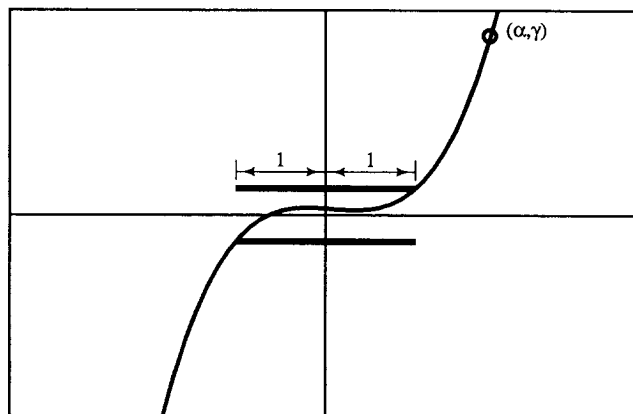


Figure 2. Univariate pipe problem

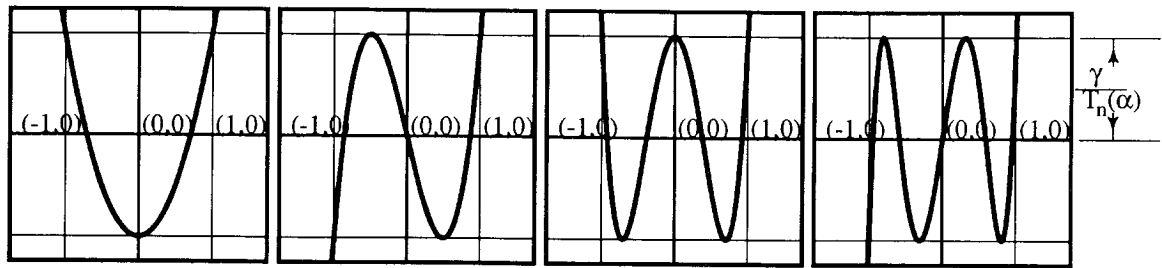


Figure 3. Plots of $\varepsilon(t)$ for $n = 2, 3, 4, 5$

least n distinct roots in $[-1, 1]$. However, $d(t)$ also has a root at $t = \alpha$, since $\varepsilon(\alpha) = \hat{\varepsilon}(\alpha) = \gamma$. Thus, $d(t)$ is a polynomial of degree of at most n with at least $n + 1$ distinct roots, and, hence, $d(t) \equiv 0$.

The solution to the univariate pipe problem for a pipe defined over an arbitrary interval $[a, b]$ is obtained by reparameterizing the Chebyshev polynomial. Denoting by

$$\text{map}(t) = \frac{2t - a - b}{b - a} \quad (14)$$

the function which maps the $[a, b]$ interval to the $[-1, 1]$ interval,

$$\varepsilon(t) = \begin{cases} \frac{\gamma}{T_n(\text{map}(\alpha))} T_n(\text{map}(t)) & \alpha \notin [a, b] \\ \gamma & \alpha \in [a, b] \end{cases} \quad (15)$$

and

$$\|\varepsilon\|_{[a,b]} = \begin{cases} \frac{|\gamma|}{|T_n(\text{map}(\alpha))|} & \alpha \notin [a, b] \\ |\gamma| & \alpha \in [a, b] \end{cases} \quad (16)$$

Multivariate pipe problem

We next extend the pipe problem to an arbitrary number of dimensions, as follows. Find the m -dimensional parametric curve $\varepsilon(t) = (\varepsilon_1(t), \varepsilon_2(t), \dots, \varepsilon_m(t))$ of degree of at most n which interpolates a single m -dimensional point $\varepsilon(\alpha) = \gamma = (\gamma_1, \gamma_2, \dots, \gamma_m)$, and which has the smallest L^2 norm

$$\|\varepsilon\|_{[-1,1]} = \max_{-1 \leq t \leq 1} (\varepsilon_1^2(t) + \varepsilon_2^2(t) + \dots + \varepsilon_m^2(t))^{1/2} \quad (17)$$

It turns out that the solution to this problem is the multivariable equivalent of the univariate pipe problem:

$$\varepsilon(t) = \begin{cases} \frac{\gamma}{T_n(\alpha)} T_n(t) & \alpha \notin [-1, 1] \\ \gamma & \alpha \in [-1, 1] \end{cases} \quad (18)$$

and

$$\|\varepsilon(t)\|_{[-1,1]} = \begin{cases} \frac{\|\gamma\|}{|T_n(\alpha)|} & \alpha \notin [-1, 1] \\ \|\gamma\| & \alpha \in [-1, 1] \end{cases} \quad (19)$$

While this is not surprising, it is not obviously true, and the following proof is not a trivial extension of the univariate case.

Proof: We again prove by contradiction. Suppose that a solution $\mathbf{q}(t) = (q_1(t), q_2(t), \dots, q_m(t))$ exists to the multivariate pipe problem for which

$$\begin{aligned} \|\varepsilon\|_{[-1,1]} &> \|\mathbf{q}\|_{[-1,1]} \\ &= \max_{-1 \leq t \leq 1} (q_1^2(t) + q_2^2(t) + \dots + q_m^2(t))^{1/2} \end{aligned} \quad (20)$$

Denote the following:

$$\begin{aligned} E(t) &= \varepsilon_1^2(t) + \varepsilon_2^2(t) + \dots + \varepsilon_m^2(t) \\ &= \frac{\gamma_1^2 + \gamma_2^2 + \dots + \gamma_m^2}{T_n^2(\alpha)} T_n^2(t) \end{aligned} \quad (21)$$

and

$$Q(t) = q_1^2(t) + q_2^2(t) + \dots + q_m^2(t) \quad (22)$$

We introduce the identity (see Reference 4, p 76)

$$T_n^2(t) = \frac{1}{2} [T_{2n}(t) + 1] \quad (23)$$

where $T_{2n} + 1$ is a shifted Chebyshev polynomial of degree $2n$. As shown in Figure 4, $T_{2n}(t) + 1$ has double roots at $t_k = \cos(((2k - 1)/2n)\pi)$, $k = 1, 2, \dots, n$. Hence, all $2n$ roots of $T_{2n} + 1$ (multiplicity counted) lie in $[-1, 1]$. $T_{2n} + 1$ has maximal values at the $n + 1$ points

$$s_k = \cos \frac{k}{n} \pi \quad k = 0, 1, 2, \dots, n \quad (24)$$

Since, by hypothesis, $\|\mathbf{q}\|_{[-1,1]} < \|\varepsilon\|_{[-1,1]}$

$$Q(s_k) < E(s_k) \quad k = 0, 1, 2, \dots, n \quad (25)$$

and

$$Q(t_k) \geq E(t_k) = 0 \quad k = 1, 2, \dots, n \quad (26)$$

If $Q(t_k) = E(t_k) = 0$, t_k is at least a double root of Q , and thus at least a double root of $Q(t) - E(t)$. If $Q(t_k) > E(t_k) = 0$, this implies that $Q(t) - E(t)$ has at least two distinct roots between s_{k-1} and s_k . Hence, $Q(t) - E(t)$ has at least $2n$ roots altogether in $[-1, 1]$. Additionally, $Q(\alpha) - E(\alpha) = 0$. Since $Q(t) - E(t)$ is a polynomial of degree of at most $2n$ which has at least $2n + 1$ roots, $Q(t) = E(t)$, and the proof is completed. \square

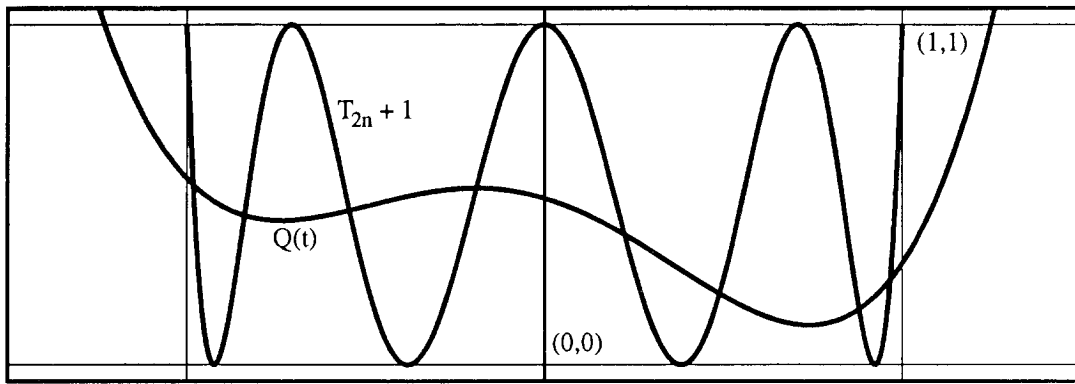


Figure 4. T_{2n+1} for $n = 4$, and sample $Q(t)$

This discussion has focused on a pipe lying in the $[-1, 1]$ interval. Obviously, for a pipe over an arbitrary interval $[a, b]$, the solution is

$$\varepsilon(t) = \begin{cases} \frac{\gamma}{T_n(\text{map}(\alpha))} T_n(\text{map}(t)) & \alpha \notin [a, b] \\ \gamma & \alpha \in [a, b] \end{cases} \quad (27)$$

and

$$\|\varepsilon(t)\|_{[a,b]} = \begin{cases} \frac{\|\gamma\|}{|T_n(\text{map}(\alpha))|} & \alpha \notin [a, b] \\ \|\gamma\| & \alpha \in [a, b] \end{cases} \quad (28)$$

PERTURBATION TO IMPOSE A LINEAR COMMON FACTOR

With the solution to the pipe problem in hand, we are now equipped to attack the problem of determining the minimum perturbation that needs to be applied to a set of polynomials as in Equation 1 to impose a linear common factor.

If $\mathbf{g}(t) = \mathbf{f}(t) + \varepsilon(t)$ (refer to Equations 1 and 2), $\mathbf{g}(\alpha) = 0$ implies that

$$\varepsilon(\alpha) = -\mathbf{f}(\alpha) \quad (29)$$

which expresses the necessary and sufficient condition for $\mathbf{g}(t)$ to have a linear common factor $t - \alpha$.

Recall that our goal is to determine the minimum (as measured by the norm $\|\varepsilon\|_{[a,b]} = \max_{a \leq t \leq b} (\varepsilon_1^2(t) + \varepsilon_2^2(t) + \dots + \varepsilon_m^2(t))^{1/2}$) perturbation $\varepsilon(t)$ that coerces the set of polynomials $\mathbf{f}(t)$ to have a common factor. We have shown that, for a specified common factor $t - \alpha$, the minimum perturbation is expressed in Equation 27. All that remains is to determine which value of α will produce the smallest perturbation.

For a given α , we have $\gamma = \varepsilon(\alpha) = -\mathbf{f}(\alpha)$. Thus, the minimum perturbation to coerce $\mathbf{g}(\alpha) = 0$ is, from Equation 27,

$$\varepsilon_\alpha(t) = \begin{cases} -\frac{\mathbf{f}(\alpha)}{T_n(\text{map}(\alpha))} T_n(\text{map}(t)) & \alpha \notin [a, b] \\ -\mathbf{f}(\alpha) & \alpha \in [a, b] \end{cases} \quad (30)$$

and the norm of the perturbation is

$$\|\varepsilon_\alpha(t)\|_{[a,b]} = \begin{cases} \frac{\|\mathbf{f}(\alpha)\|}{|T_n(\text{map}(\alpha))|} = \left(\frac{\sum_{i=1}^m f_i^2(\alpha)}{T_n^2(\text{map}(\alpha))} \right)^{1/2} & \alpha \notin [a, b] \\ \|\mathbf{f}(\alpha)\| = \left(\sum_{i=1}^m f_i^2(\alpha) \right)^{1/2} & \alpha \in [a, b] \end{cases} \quad (31)$$

We want to find the value of α for which $\|\varepsilon_\alpha\|_{[a,b]}$ (in Equation 19) is minimized. Since $\|\varepsilon_\alpha\|_{[a,b]}^2$ has minima at the same values of α as does $\|\varepsilon_\alpha\|_{[a,b]}$, it suffices to find the zeros of

$$\frac{d(\|\varepsilon_\alpha\|_{[a,b]}^2)}{d\alpha} = \begin{cases} \frac{2}{T_n^3(\text{map}(\alpha))} \sum_{i=1}^m f_i(\alpha) [(T_n(\text{map}(\alpha)) f_i'(\alpha))] & \alpha \notin [a, b] \\ -T_n'(\text{map}(\alpha)) f_i(\alpha) & \alpha \in [a, b] \\ 2 \sum_{i=1}^m f_i(\alpha) f_i'(\alpha) & \alpha \in [a, b] \end{cases} \quad (32)$$

which amounts to computing the zeros of

$$z(\alpha) = \begin{cases} \sum_{i=1}^m f_i(\alpha) [(T_n(\text{map}(\alpha)) f_i'(\alpha))] & \alpha \notin [a, b] \\ -T_n'(\text{map}(\alpha)) f_i(\alpha) & \alpha \in [a, b] \\ \sum_{i=1}^m f_i(\alpha) f_i'(\alpha) & \alpha \in [a, b] \end{cases} \quad (33)$$

Since we seek a linear common factor, α should be a real number. Thus, the solution to our problem is to compute all the real zeros of $z(\alpha)$, and to determine which of these zeros produces the smallest value of $\|\varepsilon_\alpha\|_{[a,b]}$ in Equation 31.

Example

To illustrate, consider the two polynomials shown in Figure 1:

$$f_1(t) = (t - 6)(t + 3)(t + 7) = t^3 + 4t^2 - 39t - 126 \quad (34)$$

$$f_2(t) = (t - 15)(t + 5)(t + 10) = t^3 - 175t - 750 \tag{35}$$

Their respective root sets $\{-7, -3, 6\}$ and $\{-10, -5, 15\}$ indicate nothing that is close to a common root. Choosing the interval $[a, b] = [0, 1]$, we compute

$$z(\alpha) = -224\alpha^7 + 13064\alpha^6 + 99282\alpha^5 - 2049804\alpha^4 - 20309368\alpha^3 - 35913384\alpha^2 + 53042006\alpha - 10546932$$

Note that the degree of $z(\alpha)$ is $3n - 2$, owing to the cancellation of the highest-order term in

$$[(T_n(\text{map}(\alpha))f'_i(\alpha) - T'_n(\text{map}(\alpha))f_i(\alpha)]$$

The polynomial $z(\alpha)$ has the roots

$$\{-8.69, -6.27, -5.11, 0.245, 0.746, 14.7, 62.7\} \tag{36}$$

of which $\alpha = -8.6937$ produces the smallest value of $\|\varepsilon_\alpha\|_{[a,b]}$. (A check of the roots of

$$z(\alpha) = \sum_{i=1}^m f_i(\alpha)f'_i(\alpha)$$

for the case $\alpha \in [a, b]$ produces nothing better.)

The perturbations required to coerce f_1 and f_2 to have a common root of -8.6937 work out to be

$$\varepsilon_1 = 0.0057T_3(\text{map}(t)) \tag{37}$$

$$\varepsilon_2 = -0.0046T_3(\text{map}(t)) \tag{38}$$

where, from Equation 14, $\text{map}(t) = 2t - 1$ for $[a, b] = [0, 1]$.

Scaling

These perturbations have uniform norms of 0.0057 and 0.0046. Since $\|f_1\|_{[0,1]} = 160$ and $\|f_2\|_{[0,1]} = 924$, the respective *relative* perturbations are 0.000036 and 0.0000049. This relative perturbation magnitude $\|\varepsilon_i\|/\|f_i\|$ might be, in some applications, a more meaningful measure than the absolute perturbation magnitude, since the f_i can be scaled without altering their roots.

In fact, selectively scaling the f_i prior to computing the perturbations has the effect of controlling the influence that each f_i has on the solution, since we are minimizing the absolute, rather than relative, perturbation. Thus, if a particular f_i is scaled to be larger, its resulting relative perturbation tends to be smaller.

Interval contraction

If we reduce the interval of interest, the perturbation required to induce a linear common factor tends to

Table 1. Interval contraction

Interval	α	$\ \varepsilon_1\ /\ f_1\ $	$\ \varepsilon_2\ /\ f_2\ $
[0, 1.000]	-8.69	0.0000360	0.0000049
[0, 0.500]	-8.72	5.4×10^{-6}	7.3×10^{-7}
[0, 0.250]	-8.73	7.6×10^{-7}	9.9×10^{-8}
[0, 0.125]	-8.74	1.0×10^{-7}	1.3×10^{-8}

diminish, as shown in Table 1. Our empirical experience suggests that the asymptotic rate at which the perturbation magnitude diminishes is roughly $O(2^{-m})$. In our current example, each time the interval is halved, the perturbation diminishes by roughly 1/8.

DEGREE REDUCTION OF RATIONAL CURVES

Farin and Worsley⁵ have noted that, when a rational Bézier curve is first degree elevated and then reparameterized by scaling the control-point weights w_i by c^i , where c is any constant, it is expensive to detect that the resulting curve can be represented by a curve of lower degree. What in fact happens is that, in the rational curve

$$\left(\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)}\right) \tag{39}$$

the polynomials $x(t), y(t), w(t)$ contain a common factor which can be divided out, thereby reducing the degree of the curve.

Or course, in practice, it is rare that a rational curve has an exact common factor, even within floating-point error. However, the technique in this paper provides a worthwhile capability: even if $x(t), y(t), w(t)$ have no common factor, they can possibly be perturbed by a small amount so as to impose a common factor which can be factored out, thereby resulting in an approximation with a curve of lower degree.

Consider the degree-4 rational Bézier curve in Figure 5, with control points and weight values as shown. Converted to a power basis,

$$x(t) = -8t^4 + 40t^3 - 60t^2 + 32t \tag{40}$$

$$y(t) = -24t^4 + 88t^3 - 96t^2 + 32t \tag{41}$$

$$w(t) = -6t^4 + 24t^3 - 30t^2 + 12t + 1 \tag{42}$$

which have no roots in common.

These polynomials can have a common root imposed at $t = 2.096$ by perturbing $x(t)$ by a maximum of 0.0232, $y(t)$ by a maximum of 0.0100, and $w(t)$ by a maximum

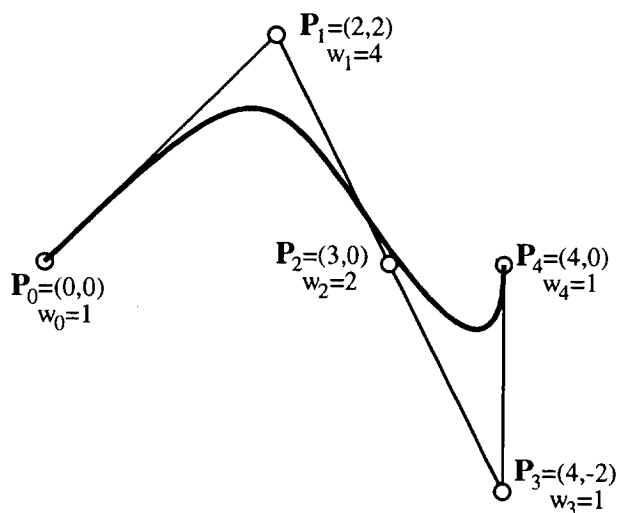


Figure 5. Degree-4 rational Bézier curve

of 0.0006. The resulting deviation from the curve in Figure 5 is within the penwidth of the plot.

What we have computed is the nearest set of polynomials which have a common factor. Note that this does not exactly represent the nearest cubic curve, since the difference of two rational curves with different denominator polynomials is not the function being minimized by our technique.

CONCLUSIONS

We have derived the minimum perturbation under which a given set of polynomials will experience a common linear factor. Even if the roots of a set of polynomials are well separated, it can happen that a relatively small perturbation will force the existence of a common root.

Repeated application of this process can produce perturbations which induce common factors of higher degree. However, these are not generally the minimum possible perturbations for inducing such higher-degree common factors. This is a subject which deserves further study.

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