Improved test for closed loops in surface intersections

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The following two theorems are proven: If two nonsingular curve segments intersect twice, if the range of normal vectors of each segment does not vary more than 90°, and if each curve is C^1 smooth, then there exists a line which is perpendicular to both curve segments simultaneously. If two nonsingular surface patches intersect in a closed loop, if the dot product between any normal vector on one surface and any other normal vector on either surface is never zero, and if the normal vector is uniquely defined at every point on each surface region, then there exists a line which is perpendicular to both surface patches simultaneously.

The second theorem is of value because it provides a criterion for eliminating a robustness limitation that arises in computing surface intersections using the marching method; namely, assuring that all branches of the intersection curve have been found. If all lines are computed that are perpendicular to both patches, and the patches are subdivided at the points at which those lines intersect the patches, then there can be no closed loops in the intersection between the set of patches obtained by subdividing the first patch and the set of patches obtained by subdividing the second patch.

computer-aided design, geometry, closed loops, surfaces, subdivision, marching

Two approaches to the problem of approximating the curve of intersection of two surface patches predominate: subdivision and marching. Subdivision-based algorithms[1] characteristically tessellate the surfaces into piecewise linear approximations and intersect the facets.

Marching, or curve-following, algorithms (see, for example, Faux and Pratt[2], Geisow[3] or Bajaj[4]) begin by finding one point on the intersection curve, and proceed to 'march' around the curve, solving three simultaneous trivariate polynomial equations for each successive point. Hybrid algorithms which use both marching and curve following have also been investigated[5].

A major limitation of marching methods is that there is no assurance that all branches of the intersection curve have been detected[6]. This paper presents a condition under which an intersection curve must contain a point on at least one patch boundary curve. This is achieved by providing a criterion for identifying a point within every closed loop in the intersection curve. Note that here a 'closed' loop is a loop whose image in the parameter space of *each patch* is closed (i.e. does not intersect any patch boundary curve).

Previous work on loop detection includes the following. Sederberg and Meyers[7] showed that if two surfaces (each C^1) intersect in a closed loop, then there exists a normal vector on one surface that is perpendicular to a tangent vector on the other surface. Sinha *et al.*[8] proves that if two surfaces (each C^1) intersect in a closed loop, then there exists a normal vector on one surface that is parallel to a normal vector on the other surface.

In this paper it is shown that if two surface patches are each C^1 and if the dot product between any normal vector on one surface and any other normal vector on either surface is never zero, then there exists a line which is perpendicular to both surfaces. Thus, the previous literature proves the existence of *parallel* normal lines, whereas this paper proves the existence of a *collinear* normal line.

An important motivation for this new theorem is that the previous loop detection tests can be very time consuming, even for some seemingly innocuous cases. The nonexistence of a collinear normal line can often be determined even though parallel normal lines may exist.

The next section discusses an analogous theorem for plane curves, and the following section discusses the second theorem dealing with closed loops of surface intersection curves.

PLANAR CURVES

Consider the two curves in Figure 1 that are everywhere differentiable and that intersect in two points. It is shown in Sinha *et al.*[8] that there must exist a normal line on one curve which is parallel to a normal line on the other curve. In fact, two such curves typically have an infinite number of pairs of parallel normal lines. Several sample normal vectors are drawn and labelled in Figure 1.

A stronger theorem is now proved about two curves which intersect in two points.

Theorem 1

If two nonsingular curve segments, $\mathbf{C}_1$ and $\mathbf{C}_2$, intersect in two points, if the tangent direction on either curve segment does not vary by more than 90°, and $\mathbf{C}_1$ and $\mathbf{C}_2$ are everywhere tangent continuous, then there exists a line which is perpendicular to both $\mathbf{C}_1$ and $\mathbf{C}_2$.

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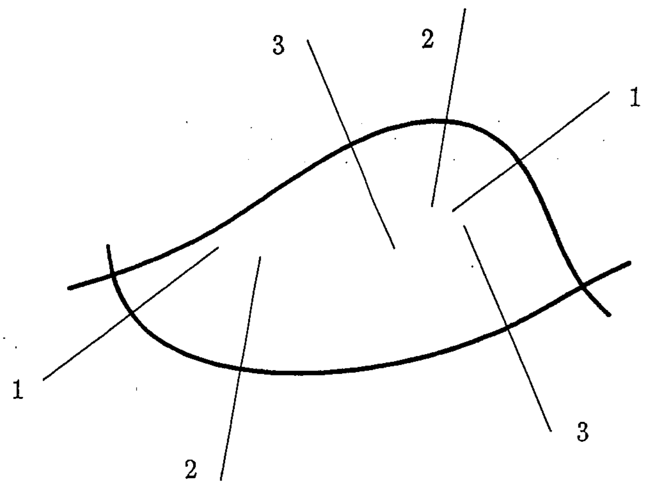

Figure 1. Intersecting curves

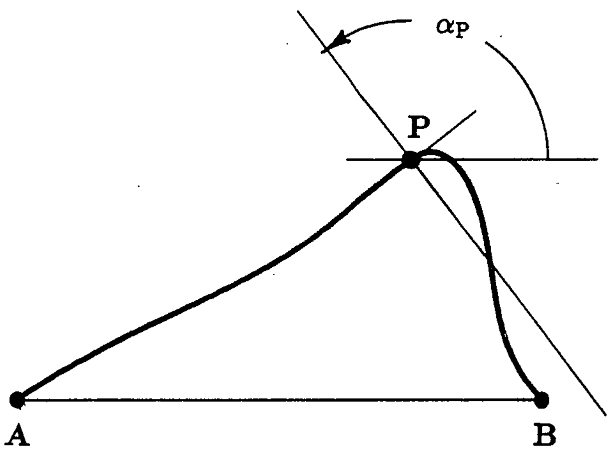

Figure 2. Normal line range

Proof. It is assumed that the two points **A** and **B** at which $\mathbf{C}_1$ and $\mathbf{C}_2$ intersect are the endpoints of the curve segments. Then the proof has three steps.

First, it is proved that the line normal to any point on $\mathbf{C}_1$ or $\mathbf{C}_2$ must pass through line segment **AB** which joins the endpoints of the curves. The convention will be adopted that the angle α of a normal line is measured counterclockwise from the line **AB**, and $0 \leqslant \alpha \leqslant 180°$. Referring to Figure 2, suppose there exists a point **P** on the curve whose normal line with angle α_P passes to the right of line segment **AB**. Considering all possible curve segments **PB**, the choice that minimizes the difference between α_P and the minimum angle on **PB** is a straight line. However, if α_{AB} is the angle of the normal to straight line **PB**, then $\alpha_P - \alpha_{AB} > 90°$, which contradicts the requirement that the curve does not bend more than 90°.

Second, let a one-to-one parametrization be imposed on line segment **AB** with parameter value $t = 0$ at point **A** and $t = 1$ at point **B**. Also, let a one-to-one parametrization be imposed on $\mathbf{C}_1$ with parameter value $s_1 = 0$ at point **A** and $s_1 = 1$ at point **B**. Similarly, parametrize $\mathbf{C}_2$ using parameter s_2. Let $\alpha(s_1)$ be the angle measured counterclockwise from **AB** to the normal line at $\mathbf{C}_1(s_1)$, and let $\alpha(s_2)$ be the angle measured counterclockwise from **AB** to the normal line at $\mathbf{C}_2(s_2)$. Finally, let $t(s_1)$ be the parameter value t at which the normal to $\mathbf{C}_1$ at $\mathbf{C}_1(s_1)$ intersects **AB** and let $t(s_2)$ be the parameter value t at which the normal to $\mathbf{C}_2$ at $\mathbf{C}_2(s_2)$ intersects **AB**.

Third, define two planar parametric curves:

$$\mathbf{P}_1(s_1) = (t(s_1), \alpha(s_1)), \qquad \mathbf{P}_2(s_2) = (t(s_2), \alpha(s_2))$$

For convention, specify that $\mathbf{C}_1$ lies above $\mathbf{C}_2$ everywhere but at the endpoints. Then, at $s_1 = s_2 = 0$, $\alpha(s_1) > \alpha(s_2)$; and at $s_1 = s_2 = 1$, $\alpha(s_1) < \alpha(s_2)$. Figure 4 shows a plot of $\mathbf{P}_1(s_1)$ and $\mathbf{P}_2(s_2)$ for the two curves in Figure 3. In this example, there are three collinear normals as shown in Figure 3. They correspond to the three intersections of $\mathbf{P}_1(s_1)$ and $\mathbf{P}_2(s_2)$ shown in Figure 4. Since $\mathbf{C}_1$ and $\mathbf{C}_2$ are C^1, $\mathbf{P}_1$ and $\mathbf{P}_2$ are C^0. Also,

$$0 \leqslant t(s_1),\ t(s_2) \leqslant 1, \qquad 0 < \alpha(s_1),\ \alpha(s_2) < 180°$$

Therefore, $\mathbf{P}_1(s_1)$ and $\mathbf{P}_2(s_2)$ must intersect. The coordinates (t, α) at which they intersect define a line that is perpendicular to both curves. Furthermore, $\mathbf{P}_1(s_1)$ and $\mathbf{P}_2(s_2)$ must intersect an odd number of times (counting multiplicity – an ordinary tangency is counted as a double intersection; a general tangency counts an even or an odd number of times if the curves stay on opposite sides or 'cross over' at the point of tangency, respectively). Thus, there are always an odd number of collinear normal lines, counting multiplicity.

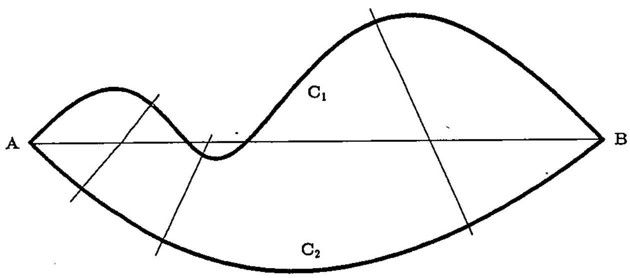

Figure 3. Two intersecting curves

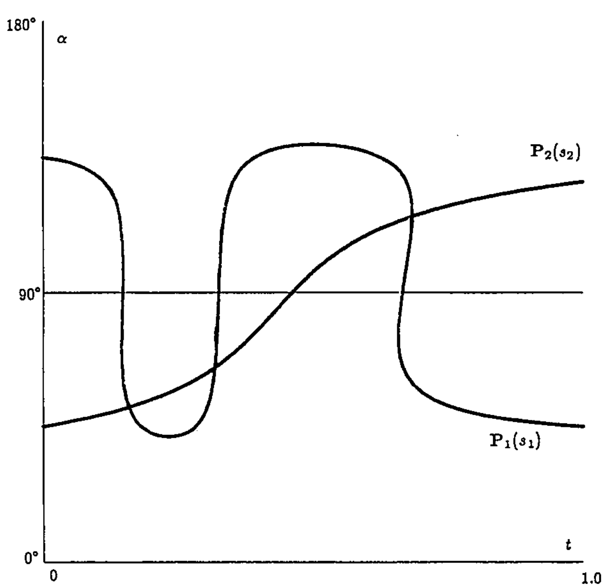

Figure 4. $\mathbf{P}_1(s_1)$ and $\mathbf{P}_2(s_2)$

CLOSED LOOPS OF SURFACE INTERSECTIONS

A slightly different approach is taken in proving the analogous theorem for surfaces.

Theorem 2
If two nonsingular surface patches, $\mathbf{S}_1$ and $\mathbf{S}_2$, intersect in a closed loop $\mathbf{C}$, then there exists a line that is perpendicular to both $\mathbf{S}_1$ and $\mathbf{S}_2$ if the following conditions are met:

- The dot product of any two normal vectors (on the same patch or on different patches) is never zero. This means that the total range of normal directions *for both patches considered simultaneously* cannot deviate more than 90°. This condition is more restrictive than in Theorem 1.
- $\mathbf{S}_1$ and $\mathbf{S}_2$ are everywhere tangent continuous.

Proof. In this proof, the Brouwer fixed point theorem from topology is used in an essential way. To improve the readability of this paper, the statement of the Brouwer fixed point theorem is reproduced here.

Brouwer fixed point theorem (Munkres[9], Chap. 8, Theorem 10.2)
Every continuous mapping from the closed disc of radius 1 in the plane to itself has a fixed point.

This theorem applies as well to any region that admits a parametrization by the unit disc. In the situation considered here, the region in question is a subregion of a surface patch.

Assume the existence of a closed loop, and furthermore assume that there are no points at which the two surfaces are tangent. If indeed there exists a point of tangency, then a collinear normal line occurs at that point.

- *Step 1.* Any line L_1 perpendicular to $\mathbf{S}_1$ intersects $\mathbf{S}_1$ exactly once, else condition 1 is violated. To see this, pass a plane T through L_1 and assume L_1 meets $\mathbf{S}_1$ in two points p and q which are joined by both the line L_1 and an arc A of the curve $T \cap \mathbf{S}_1$. By the mean value theorem of calculus, there is a point of A at which the tangent is parallel to L_1, hence the normal is perpendicular to L_1. This is a contradiction.
- *Step 2.* Any line L_1 perpendicular to $\mathbf{S}_1$ at a point p_1 interior to $\mathbf{C}$ must intersect $\mathbf{S}_2$ exactly once at a point p_2 interior to $\mathbf{C}$. This follows by combining Step 1 with the argument in the first step of the proof of Theorem 1. Denote this mapping $p_2 = N_1(p_1)$. Likewise, any line L_2 perpendicular to $\mathbf{S}_2$ at a point p_2 interior to $\mathbf{C}$ must intersect $\mathbf{S}_1$ exactly once at a point p_1 interior to $\mathbf{C}$. Denote this mapping $p_1 = N_2(p_2)$. Note that $p_1 = N_2(N_1(p_1))$ only if either p_1 lies on a collinear normal line, or if p_1 lies on $\mathbf{C}$.
- *Step 3.* The only way $N_1(p_1)$ lies on $\mathbf{C}$ is if $p_1 = N_1(p_1)$.
- *Step 4.* Let P_0 be the region in S_1 bounded by $\mathbf{C}$. Consider the map f from P_0 to P_0 defined by $f(p) = N_2(N_1(p))$. Impose a continuous (and continuously invertible) one-to-one parameterization on P_0, such that the domain of parameter values is the unit disc. It is desired to apply the Brouwer fixed point theorem discussed previously to f to guarantee a fixed point, but this is premature; a fixed point of f on $\mathbf{C}$ will not be conclusive. For each small ε let $\mathbf{C}_\varepsilon$ be the curve in P_0 offset from $\mathbf{C}$ by the distance ε, and let P_ε be the region it contains. Note that for each point p near $\mathbf{C}$, f(p) is further from $\mathbf{C}$ than p is. Let m be such that for all points p within a distance m of $\mathbf{C}$, f(p) is further from $\mathbf{C}$ than p is. This implies that for all positive $\varepsilon < m$, $f(P_\varepsilon - P_m) \subset P_\varepsilon$. Now $f(P_m) \subset P_0$ does not intersect $\mathbf{C}$ by Step 1, so is contained in some $P_{m'}$ with $0 < m' < m$. Now f maps $P_{m'}$ to itself: $f(P_{m'}) = f(P_{m'} - P_m) \cup f(P_m) \subset P_{m'} \cup P_{m'} = P_{m'}$.
- *Step 5.* Since $P_{m'}$ can be parameterized by a disc, the Brouwer fixed point theorem guarantees the existence of a fixed point of f in $P_{m'}$. Since this point is necessarily an interior point of P_0, the theorem is proved.

IMPLEMENTATION

One way to capitalize on the collinear normal test is as follows. If two patches do *not* contain a collinear normal (and do not turn more than 90°), then there is no closed loop. Denoting the two surfaces by $\mathbf{P}(s, t)$ and $\mathbf{Q}(u, v)$, a collinear normal line satisfies all of the following four equations:

$$\mathbf{P}_s \times \mathbf{P}_t \cdot \mathbf{Q}_u = 0 \tag{1}$$

$$\mathbf{P}_s \times \mathbf{P}_t \cdot \mathbf{Q}_v = 0 \tag{2}$$

$$(\mathbf{P} - \mathbf{Q}) \cdot \mathbf{P}_s = 0 \tag{3}$$

$$(\mathbf{P} - \mathbf{Q}) \cdot \mathbf{P}_t = 0 \tag{4}$$

An efficient way to determine the nonexistence of collinear normals is to use interval arithmetic[10]. The bivariate polynomials $\mathbf{P}, \mathbf{Q}, \mathbf{P}_s, \mathbf{P}_t, \mathbf{Q}_u$ and $\mathbf{Q}_v$ are replaced by interval constants:

$$\mathbf{P} = ([x_{min}, x_{max}], [y_{min}, y_{max}], [z_{min}, z_{max}])$$

and so on, which simply amount to min–max bounding boxes. When each of the vector equations (1)–(4) is expanded using interval arithmetic (see Mudur and Koparkar[10]), the result is an interval expression

$$[a, b] = 0$$

where the interval $[a, b]$ bounds the value of the equation for all values of s, t, u, v in the unit interval. Therefore, if $0 \notin [a, b]$ for any of the equations (1)–(4), a collinear normal does not exist for $\mathbf{P}$ and $\mathbf{Q}$. If, on the other hand, $0 \in [a, b]$ for all four equations, the surfaces are subdivided and the test is applied again. Eventually a set of subdivided patches that intersect is arrived at, but with no closed loops.

This test can be improved by changing the coordinate system such that one of the patches tends to be parallel to a coordinate plane, thereby diminishing the interval width.

The authors are currently studying algorithms for robustly and efficiently computing collinear normals (as opposed to determining their nonexistence). This can be accomplished now with linear convergence by simply subdividing until two sufficiently small patches are obtained that satisfy equations (1)–(4). The authors are optimistic that a quadratically convergent method based on interval Newton iteration should be possible. If this succeeds, then a loop can be split by simply subdividing one of the patches at the point through which the collinear normal line passes. Meanwhile, the interval algorithm works, except that point tangencies get very expensive.

Note that the collinear normal theorem holds true for implicit surfaces as well as parametric surfaces. However, the detection of collinear normals for a pair of implicit surfaces is a difficult problem.

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