Chapter 12

Approximating Rational Curves Using Polynomial Curves

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12.1. Introduction

Rational Bézier curves are defined by the equation

$$\mathbf{R}(t) = \frac{\sum_{i=0}^{r} w_i \mathbf{R}_i B_i^r(t)}{\sum_{i=0}^{r} w_i B_i^r(t)}, \tag{12.1}$$

where $B_i^r(t) = \binom{r}{i}(1-t)^{r-i}t^i$. The values $\mathbf{R}_i$ are called control points and the w_i are the control point weights [1]. A polynomial Bézier curve is defined

$$\mathbf{P}(t) = \sum_{i=0}^{p} \mathbf{P}_i B_i^p(t). \tag{12.2}$$

It is well known that any polynomial Bézier curve can be represented as a rational Bézier curve by letting $\mathbf{R}_i = \mathbf{P}_i$ and by setting all w_i to the same nonzero value w:

$$\mathbf{R}(t) = \frac{\sum_{i=0}^{r} w\mathbf{P}_i B_i^r(t)}{\sum_{i=0}^{r} w B_i^r(t)} = \frac{w\sum_{i=0}^{r} \mathbf{P}_i B_i^r(t)}{w\sum_{i=0}^{r} B_i^r(t)} = \frac{w\sum_{i=0}^{r} \mathbf{P}_i B_i^r(t)}{w} = \mathbf{P}(t).$$

Alternately [2], one can express a polynomial Bézier curve in rational form by using the bilinear reparameterization

$$t(u) = \frac{cu}{(1-u)+cu}$$

for which $t(0) = 0$ and $t(1) = 1$ for any value c. Then,

$$\begin{aligned}
\mathbf{P}(u) &= \sum_{i=0}^{p} \mathbf{P}_i B_i^p\left(\frac{cu}{(1-u)+cu}\right) \\
&= \sum_{i=0}^{p} \mathbf{P}_i \binom{p}{i}\left(1 - \frac{cu}{(1-u)+cu}\right)^{p-i}\left(\frac{cu}{(1-u)+cu}\right)^{i} \\
&= \sum_{i=0}^{p} \mathbf{P}_i \binom{p}{i}\left(\frac{1-u}{(1-u)+cu}\right)^{p-i}\left(\frac{cu}{(1-u)+cu}\right)^{i}
\end{aligned}$$

$$= \frac{\sum_{i=0}^{p} \mathbf{P}_i \binom{p}{i} (1-u)^{p-i} (cu)^i}{[(1-u)+cu]^p}$$

$$= \frac{\sum_{i=0}^{p} c^i \mathbf{P}_i B_i^p(u)}{\sum_{i=0}^{p} c^i B_i^p(u)},$$

which can be interpreted as a rational Bézier curve for which $\mathbf{R}_i = \mathbf{P}_i$ and $w_i = c^i$. If $\mathbf{P}(t)$ is degree elevated before the reparameterization $t(u) = cu/((1-u)+cu)$ is imposed, a more complicated representation of the polynomial curve as a rational curve results [3].

This paper addresses the reverse problem of approximating an arbitrary rational Bézier curve $\mathbf{R}(t)$ with a polynomial Bézier curve $\mathbf{P}(t)$. Our approach is based on the observation that any rational Bézier curve can be expressed as a polynomial Bézier curve with one *moving control point* which is itself a rational Bézier curve. We will refer to a polynomial curve with a moving control point as a *hybrid* curve. A hybrid curve of degree p with moving control point m is expressed

$$\mathbf{H}(t) = \sum_{i=0, i \neq m}^{p} \mathbf{P}_i B_i^p(t) + \frac{\sum_{i=0}^{r} w_i \mathbf{M}_i B_i^r(t)}{\sum_{i=0}^{r} w_i B_i^r(t)} B_m^p(t) = \mathbf{R}(t), \tag{12.3}$$

where $\mathbf{M}(t) = \frac{\sum_{i=0}^{r} w_i \mathbf{M}_i B_i^r(t)}{\sum_{i=0}^{r} w_i B_i^r(t)}$ is the moving control point, and w_i are the same weights as in (12.1). The idea of a moving control point originates with Ball [4], who formulated a rational cubic curve as a degree 2 Bézier curve with a middle control point in motion.

Figure 12.1 shows a sample cubic rational Bézier curve and Fig. 12.2 shows a quadratic hybrid curve with control points $\mathbf{P}_0$, $\mathbf{M}(t)$, $\mathbf{P}_2$, where the moving control point $\mathbf{M}(t)$ is a cubic Bézier curve. In order to evaluate this hybrid curve at some value t, first evaluate $\mathbf{M}(t)$ and then evaluate at the same parameter value t the polynomial quadratic curve with control points $\mathbf{P}_0$, $\mathbf{M}(t)$, $\mathbf{P}_2$. Figure 12.3 shows the hybrid curve evaluated at $t = .5$.

A rational curve can be represented as a hybrid curve of any degree, and any control point can be identified as moving. The degree of the moving control point is always the degree of the initial rational curve. Figures 12.4–12.6 show the curve in Fig. 12.1 represented as hybrids of degree 4, 6, and 8. In each case the middle control point is moving.

Note that the hybrid curve is an exact representation of, not an approximation of, the rational curve.

The development of hybrid curves is discussed in §12.2. Section 12.3 discusses approximation.

12.2. Creating Hybrid Curves

The most direct method for solving for a hybrid representation of a given rational curve is through algebraic manipulation. We can freely choose the degree p of the hybrid curve, and also can specify which control point m is

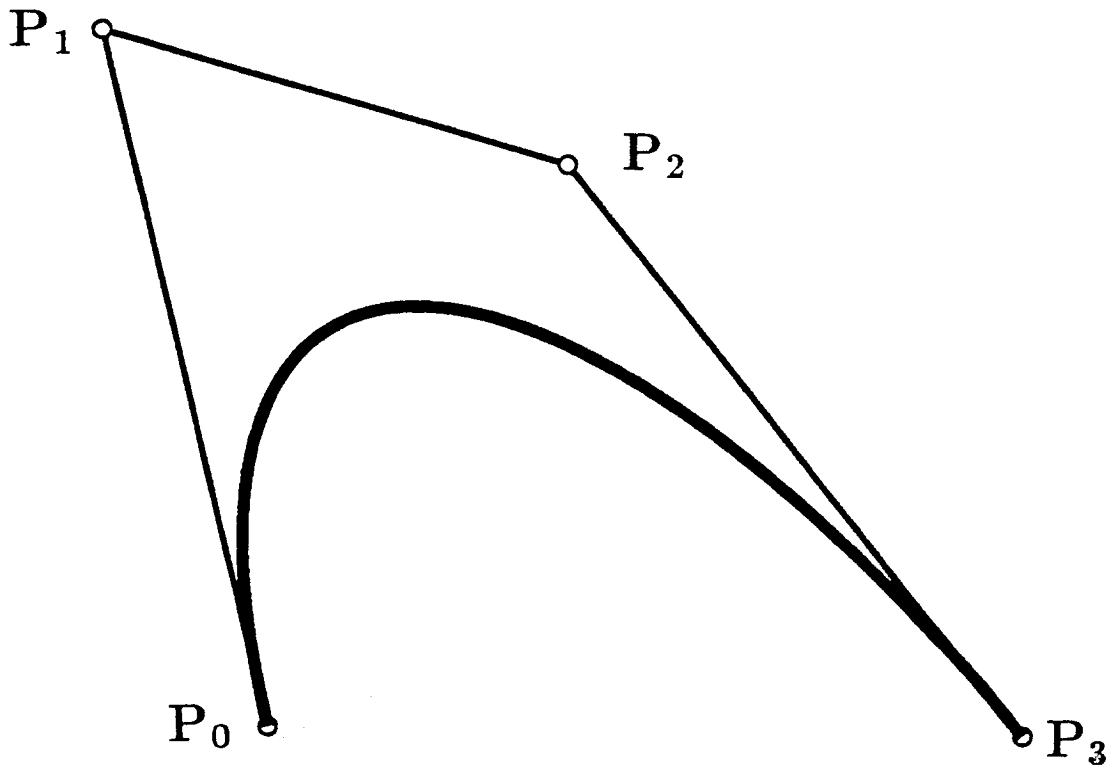

FIG. 12.1. *Cubic rational Bézier curve.*

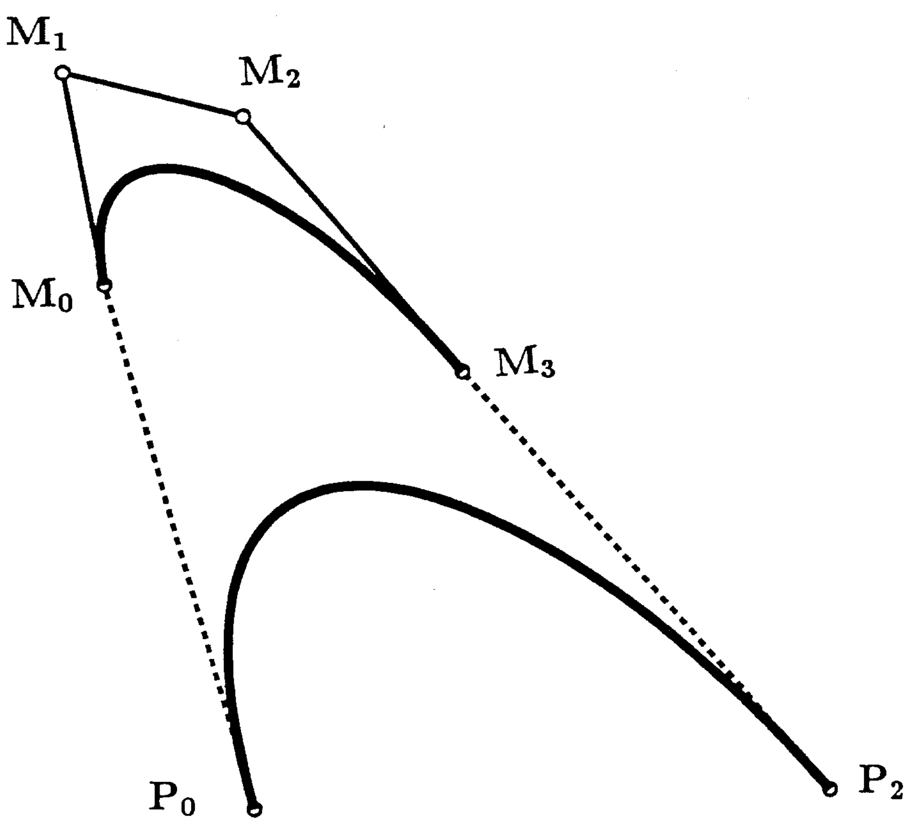

FIG. 12.2. *Quadratic hybrid curve.*

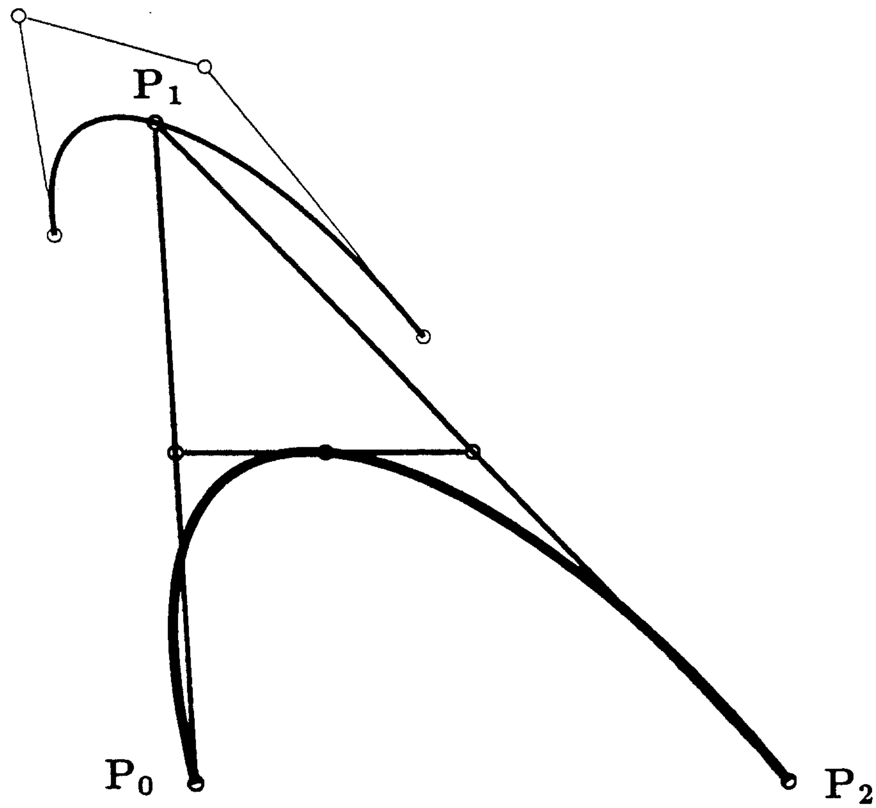

FIG. 12.3. *Evaluating at $t = .5$.*

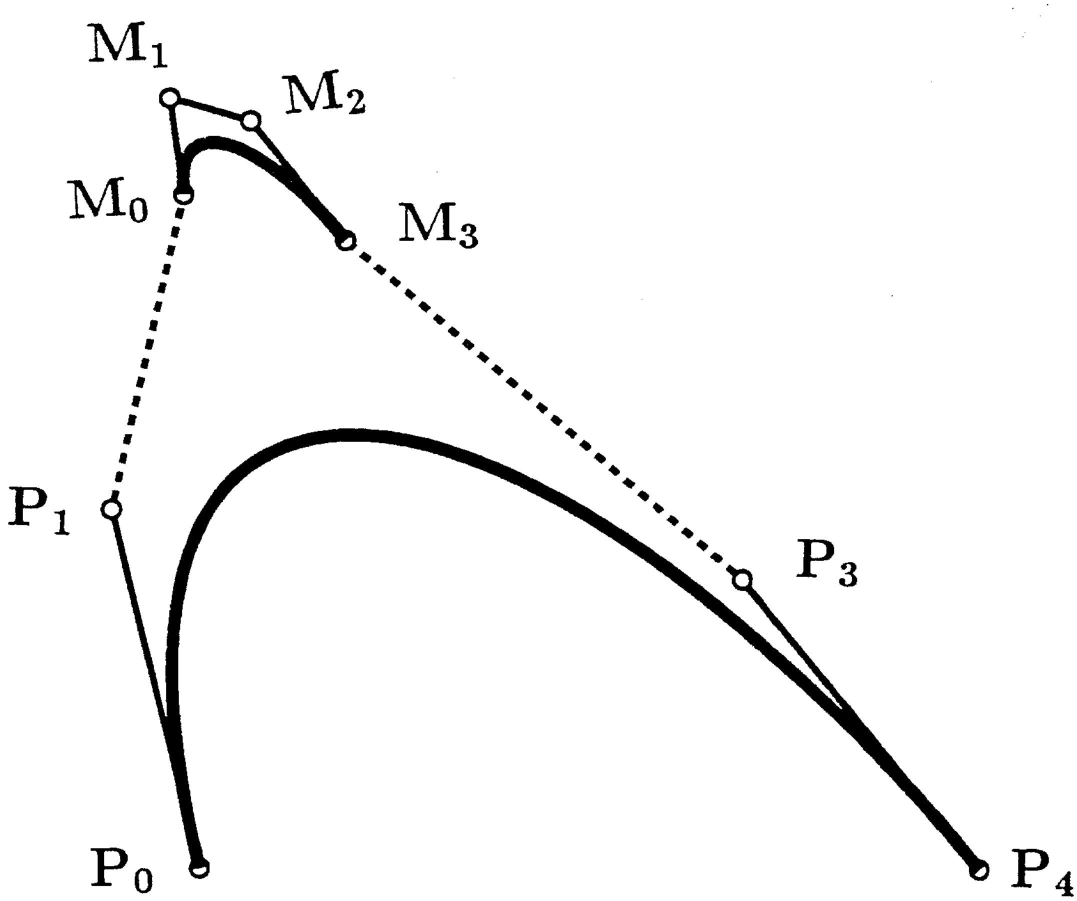

FIG. 12.4. *Degree 4 hybrid.*

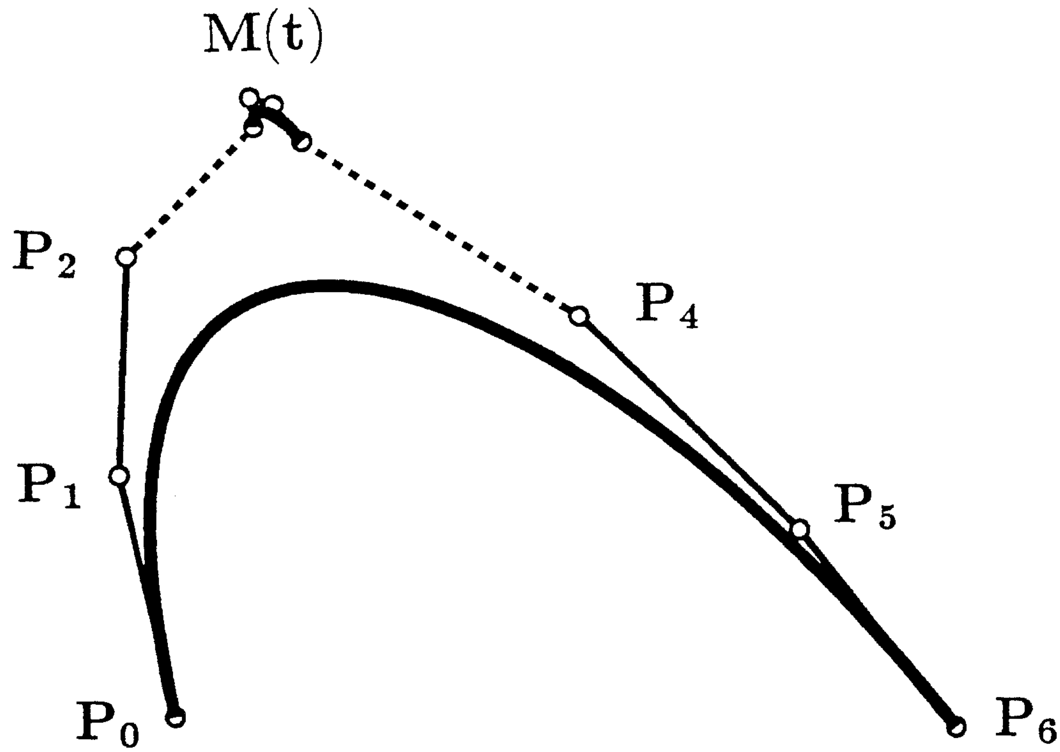

FIG. 12.5. *Degree 6 hybrid.*

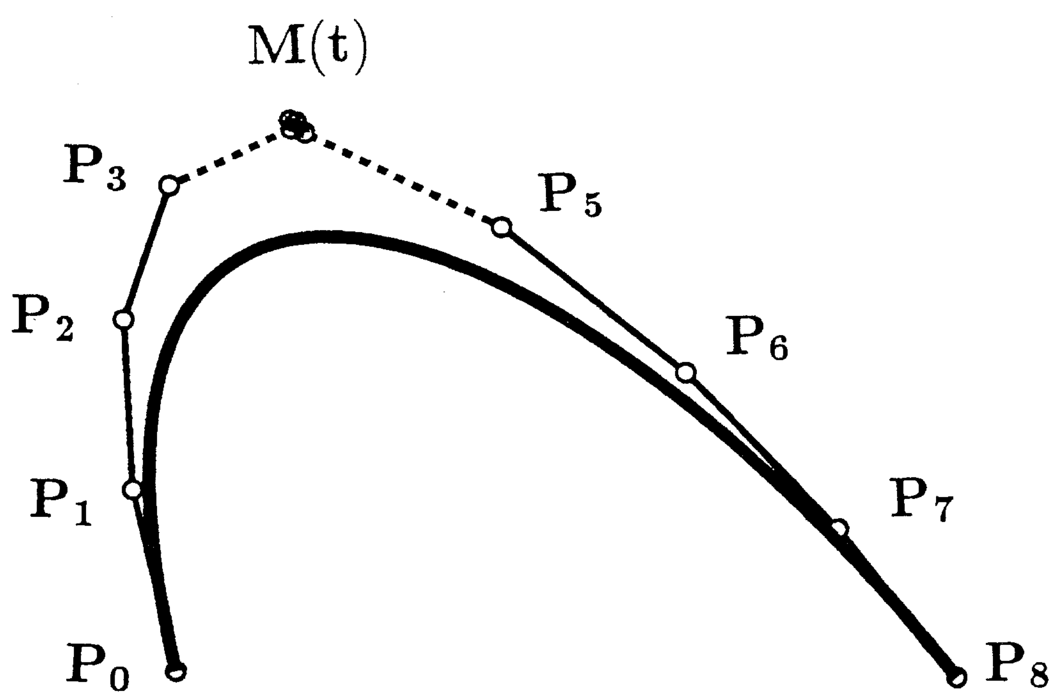

FIG. 12.6. *Degree 8 hybrid.*

the moving control point. We then solve for the $\mathbf{P}_i$ and the $\mathbf{M}_i$ in (12.3) by equating (12.1) and (12.3):

$$\begin{aligned}
\mathbf{R}(t) = \frac{\sum_{i=0}^{r} w_i \mathbf{R}_i B_i^r(t)}{\sum_{i=0}^{r} w_i B_i^r(t)} &= \sum_{i=0, i\neq m}^{p} \mathbf{P}_i B_i^p(t) \\
&\quad + \frac{\sum_{i=0}^{r} w_i \mathbf{M}_i B_i^r(t)}{\sum_{i=0}^{r} w_i B_i^r(t)} B_m^p(t), \\
\sum_{i=0}^{r} w_i \mathbf{R}_i B_i^r(t) &= \sum_{i=0}^{r} w_i B_i^r(t) \sum_{i=0, i\neq m}^{p} \mathbf{P}_i B_i^p(t) \\
&\quad + B_m^p(t) \sum_{i=0}^{r} w_i \mathbf{M}_i B_i^r(t), \\
\sum_{i=0}^{r+p} \frac{\sum_{j+k=i} w_j \mathbf{R}_j \binom{r}{j}\binom{p}{k}}{\binom{r+p}{i}} B_i^{r+p}(t) &= \sum_{i=0}^{r+p} \frac{\sum_{j+k=i, k\neq m} w_j \mathbf{P}_k \binom{r}{j}\binom{p}{k}}{\binom{r+p}{i}} B_i^{r+p}(t) \\
&\quad + \sum_{i=m}^{m+r} \frac{w_{i-m} \mathbf{M}_{i-m} \binom{p}{m}\binom{r}{i-m}}{\binom{r+p}{i}} B_i^{r+p}(t).
\end{aligned}$$

Thus, for $i = 0, \cdots, m-1, m+r+1, \cdots, r+p$,

$$\sum_{j+k=i} \binom{r}{j} \binom{p}{k} w_j [\mathbf{P}_k - \mathbf{R}_j] = 0 \tag{12.4}$$

and for $i = m, \cdots, m+r$,

$$\mathbf{M}_{i-m} = \frac{\sum_{j+k=i} \binom{r}{j}\binom{p}{k} w_j [\mathbf{R}_j - \mathbf{P}_k]}{w_{i-m} \binom{p}{m}\binom{r}{i-m}}. \tag{12.5}$$

These equations allow us to solve in the following order, $\mathbf{P}_0, \cdots, \mathbf{P}_{m-1}$; $\mathbf{P}_p, \cdots, \mathbf{P}_{m+1}$; and $\mathbf{M}_0, \cdots, \mathbf{M}_r$. From (12.4), for $i = 0, \cdots, m-1$,

$$\begin{aligned}
\mathbf{P}_0 &= \mathbf{R}_0, \\
\mathbf{P}_1 &= \mathbf{R}_0 + \frac{w_1 r [\mathbf{R}_1 - \mathbf{P}_0]}{w_0 p}, \\
\mathbf{P}_2 &= \mathbf{R}_0 + \frac{w_1 r p [\mathbf{R}_1 - \mathbf{P}_1] + w_2 \binom{r}{2} [\mathbf{R}_2 - \mathbf{P}_0]}{w_0 \binom{p}{2}}, \\
\mathbf{P}_i &= \mathbf{R}_0 + \frac{\sum_{j=1}^{\min(r,i)} \binom{r}{j}\binom{p}{i-j} w_j [\mathbf{R}_j - \mathbf{P}_{i-j}]}{w_0 \binom{p}{i}}. \qquad (12.6)
\end{aligned}$$

For $i = p+r, \cdots, m+1+r$,

$$\begin{aligned}
\mathbf{P}_p &= \mathbf{R}_r, \\
\mathbf{P}_{p-1} &= \mathbf{R}_r + \frac{w_{r-1} r [\mathbf{R}_{r-1} - \mathbf{P}_p]}{w_r p}, \\
\mathbf{P}_{i-r} &= \mathbf{R}_r + \frac{\sum_{j=\max(0,i-p)}^{r-1} \binom{r}{j}\binom{p}{i-j} w_j [\mathbf{R}_j - \mathbf{P}_{i-j}]}{w_r \binom{p}{i-r}}. \qquad (12.7)
\end{aligned}$$

From (12.5),

$$\text{(12.8)} \quad \mathbf{M}_i = \sum_{j=\max(0,i+m-p)}^{\min(i+m,r)} \frac{\binom{r}{j}\binom{p}{i+m-j} w_j[\mathbf{R}_j - \mathbf{P}_{i+m-j}]}{w_i\binom{p}{m}\binom{r}{i}}, \qquad i = 0, \cdots, r.$$

12.3. Polynomial Approximation

The hybrid representation $\mathbf{H}(t)$ of a rational curve $\mathbf{R}(t)$ leads directly to a polynomial approximation $\mathbf{P}(t)$; simply replace the moving control point with a stationary control point $\mathbf{P}_m$.

An intelligent choice is the midpoint of the $x - y$ min-max box which bounds the moving control point. Denote the half widths of that bounding box $\Delta = (\delta x, \delta y)$. A bound on the error with which $\mathbf{P}$ approximates $\mathbf{H}$ (and therefore with which it approximates $\mathbf{R}$) is derived as follows:

$$\text{(12.9)} \qquad \mathbf{P}(t) - \mathbf{H}(t) = (\mathbf{M}(t) - \mathbf{P}_m)B^p_m(t).$$

Since $|\mathbf{M}(t) - \mathbf{P}_m)| \le |\Delta|$ and since the Bernstein polynomial $B^p_m(t)$ assumes its maximum value at $t = \frac{m}{p}$ of $\binom{p}{m}(1-\frac{m}{p})^{p-m}(\frac{m}{p})^m$,

$$\text{(12.10)} \qquad |\mathbf{P}(t) - \mathbf{H}(t)| \le |\Delta|\binom{p}{m}\left(1-\frac{m}{p}\right)^{p-m}\left(\frac{m}{p}\right)^m.$$

For practical applications, it generally works best if p is an even number, and $m = p/2$. In this case,

$$\text{(12.11)} \qquad |\mathbf{P}(t) - \mathbf{H}(t)| \le |\Delta|\binom{p}{p/2}\frac{1}{2^p}.$$

The hybrid curve method lends itself directly to piecewise Hermite approximation of rational curves. Note that the hybrid curve (and hence the polynomial approximation) matches $p/2$ derivatives at each end point with the rational curve. This can be seen by differentiating (12.9) k times and noting that

$$\text{(12.12)} \qquad \mathbf{P}^{(k)}(0) - \mathbf{H}^{(k)}(0) = 0, \qquad k = 0, \cdots, m-1,$$

$$\text{(12.13)} \qquad \mathbf{P}^{(k)}(1) - \mathbf{H}^{(k)}(1) = 0, \qquad k = m+1, \cdots, p.$$

To approximate a given rational curve with piecewise $C^{p/2}$ Hermite curves, first compute the error bound that is generated by performing the approximation with a single curve. If that error is too large, subdivide the curve in half and approximate using two curves. Then recursively check the error on the new approximating curve segments. In the limit, replacing a single approximating curve with two approximating curves of equal parameter length will reduce the error by a factor of 2^{-p} (see [5], p. 67).

For example, Fig. 12.7 shows a quartic rational curve expressed using a single quartic hybrid curve. If we replace the moving control point with a single control point, the error bound in the polynomial approximation is 0.24, where the maximum horizontal width of the curve is 1. Figure 12.8 shows the same quartic rational curve expressed using two quartic hybrid curves. In this case, when we replace the moving control points with stationary control points, the resulting piecewise Hermite approximation has an error bound of 0.02.

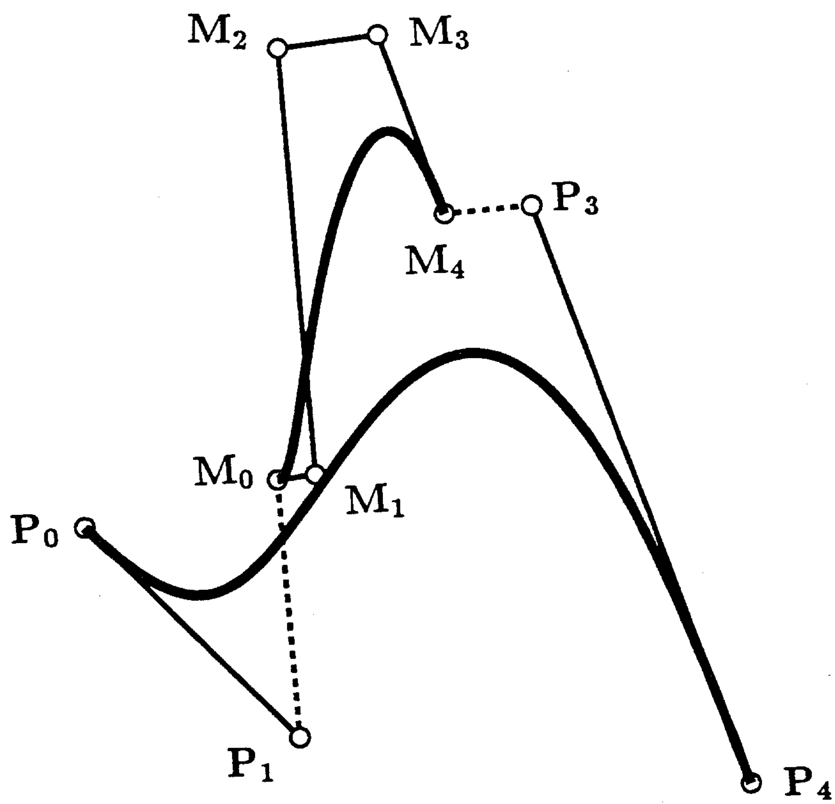

FIG. 12.7. *Quartic hybrid curve.*

As a final example, we computed the hybrid representation of a semicircle expressed exactly as a rational cubic Bézier curve with control points $(-1,0)$, $(-1,2)$, $(1,2)$, and $(1,0)$ and weights $1, 1/3, 1/3, 1$. The error bounds in Table 12.1 were computed for approximating curves of degree 2, 4, 6, and 8 using one, two, four, and eight curve segments in the approximation. Clearly, this is not an optimal approximation for a semicircle, but it does illustrate how this method performs.

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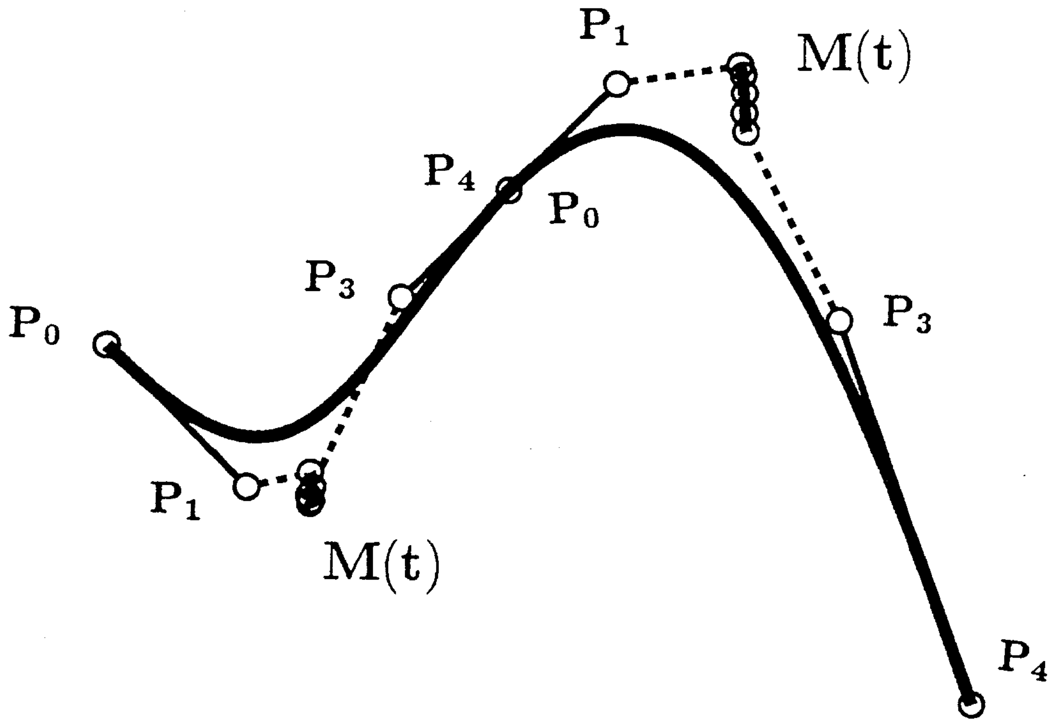

FIG. 12.8. *Two quartic hybrid curves.*

TABLE 12.1
Error bounds in approximating a semicircle.

Degree	Number of Segments			
	1	2	4	8
2	0.5	0.125	0.01	0.00325
4	0.25	0.016	0.0014	4.1×10^{-5}
6	0.125	0.0039	7.5×10^{-5}	7.0×10^{-7}
8	0.0625	4.9×10^{-4}	3.2×10^{-6}	1.2×10^{-8}

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