Short Communication

Gaussian and mean curvatures of rational Bézier patches

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Abstract

This note derives formulae for Gaussian and mean curvatures for tensor-product and triangular rational Bézier patches in terms of the respective control meshes. These formulae provide more geometric intuition than the generic formulae from differential geometry.

Key words: Rational Bézier patches, Gaussian curvature, Mean curvature

1 Introduction

The rational Bézier form is a common representation in CAGD and computer graphics. A Bézier control polygon or mesh not only mimics the shape of the curve or surface, but also conveys much geometric insight. For example, the curvature of a degree n rational Bézier curve with control points $\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_n$ and weights $\omega_0, \omega_1, \cdots, \omega_n$, at the starting point $\mathbf{P}_0$ is given by (Farin, 1990)

$$k = \frac{n-1}{n} \frac{\omega_0 \omega_2}{\omega_1^2} \frac{h}{a^2} \tag{1}$$

where a is the length of edge $\mathbf{P}_0\mathbf{P}_1$, and h is the distance of $\mathbf{P}_2$ to the tangent spanned by $\mathbf{P}_0$ and $\mathbf{P}_1$ (see Figure 1). This formula is more intuitive than the one given by classic differential geometry, and is also easier to compute, especially in the rational case.

This note derives similar formulae for Gaussian and mean curvatures of rational Bézier patches. The derived formulae are expressed in terms of simple geometric quantities of the control mesh.

2 Curvatures of tensor-product rational Bézier surfaces

Suppose a degree $n \times m$ rational Bézier patch is defined by

$$\mathbf{r}(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} \omega_{ij}\mathbf{P}_{ij}B_i^n(u)B_j^m(v)}{\sum_{i=0}^{n}\sum_{j=0}^{m} \omega_{ij}B_i^n(u)B_j^m(v)} \tag{2}$$

where $B_i^k(t) = \binom{k}{i}(1-t)^{k-i}t^i$ are Bernstein polynomials; $\mathbf{P}_{ij}$ are the control points, forming a control mesh; and w_{ij} are the weights. When all the weights are the same, the patch reduces to a polynomial surface. We derive the curvature formulae at the bottom-left corner $(u,v) = (0,0)$.

By differential geometry (Do Carmo, 1976), Gaussian curvature K_g and mean curvature K_m can be computed by the following formulas

$$K_g = \frac{LN - M^2}{EG - F^2}, \tag{3}$$

$$K_m = \frac{1}{2}\,\frac{LG - 2MF + NE}{EG - F^2} \tag{4}$$

where E, F, G are the coefficients of the first fundamental form, i.e.,

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v, \tag{5}$$

L, M, N are the coefficients of the second fundamental form, i.e.,

$$L = \mathbf{r}_{uu} \cdot \mathbf{n}, \quad M = \mathbf{r}_{uv} \cdot \mathbf{n}, \quad N = \mathbf{r}_{vv} \cdot \mathbf{n}, \tag{6}$$

and $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v / \|\mathbf{r}_u \times \mathbf{r}_v\|$ is the unit normal vector.

For the rational Bézier patch (2), we introduce some notations as follows (see Figure 2):

$$\begin{aligned} &\mathbf{a} = \mathbf{P}_{10} - \mathbf{P}_{00}, \quad \mathbf{b} = \mathbf{P}_{01} - \mathbf{P}_{00}, \quad \mathbf{c} = \mathbf{P}_{20} - \mathbf{P}_{10}, \\ &\mathbf{d} = \mathbf{P}_{02} - \mathbf{P}_{01}, \quad \mathbf{e} = \mathbf{P}_{11} - \mathbf{P}_{10}, \quad \mathbf{f} = \mathbf{P}_{11} - \mathbf{P}_{01}. \end{aligned} \tag{7}$$

Furthermore, let S denote the area of the triangle $\mathbf{P}_{00}\mathbf{P}_{10}\mathbf{P}_{01}$, θ be the angle between vectors $\mathbf{P}_{00}\mathbf{P}_{10}$ and $\mathbf{P}_{00}\mathbf{P}_{01}$, and h_{ij} be the signed distance from $\mathbf{P}_{ij}$ to the plane spanned by $\mathbf{P}_{00}, \mathbf{P}_{10}$ and $\mathbf{P}_{01}$. If $\mathbf{P}_{ij}$ lies on the side of the plane

$\mathbf{P}_{00}\mathbf{P}_{10}\mathbf{P}_{01}$ with the direction $\mathbf{P}_{00}\mathbf{P}_{10} \times \mathbf{P}_{00}\mathbf{P}_{01}$, h_{ij} is positive. Otherwise, h_{ij} is negative.

It is easy to check that at $(u, v) = (0, 0)$, we have

$$\mathbf{r}_u(0,0) = n\frac{\omega_{10}}{\omega_{00}}\mathbf{a}, \qquad \mathbf{r}_v(0,0) = m\frac{\omega_{01}}{\omega_{00}}\mathbf{b}. \tag{8}$$

For notational simplicity, in the following where there is no ambiguity, we omit the parameter values $(0, 0)$. Thus

$$E = n^2\left(\frac{\omega_{10}}{\omega_{00}}\right)^2 \mathbf{a}\cdot\mathbf{a}, \quad F = nm\left(\frac{\omega_{10}\omega_{01}}{\omega_{00}^2}\right)\mathbf{a}\cdot\mathbf{b}, \quad G = m^2\left(\frac{\omega_{01}}{\omega_{00}}\right)^2 \mathbf{b}\cdot\mathbf{b}. \tag{9}$$

Note that for a rational surface $\mathbf{r}(u, v) = \mathbf{R}(u, v)/\omega(u, v)$, the first partial derivative $\mathbf{r}_u(u, v) = \dfrac{\mathbf{R}_u}{\omega} - \mathbf{r}\dfrac{\omega_u}{\omega}$, and the second partial derivative $\mathbf{r}_{uu} = \dfrac{\mathbf{R}_{uu} - \mathbf{r}\omega_{uu}}{\omega} - \dfrac{\mathbf{R}_u\omega_u - \mathbf{r}\omega_u^2}{\omega^2} - \mathbf{r}_u\dfrac{\omega_u}{\omega}$. Applying these to the rational Bézier patch (2) and letting $(u, v) = (0, 0)$ lead to

$$\mathbf{r}_{uu} = n(n-1)\frac{\omega_{20}}{\omega_{00}}\mathbf{c} + (\cdots)\mathbf{a}$$

where $(\cdots)$ denotes a rather complicated expression that we will not need to be concerned with since subsequent dotting with $\mathbf{n}$ will cause it to vanish. Similarly, we can obtain

$$\mathbf{r}_{uv} = nm\frac{\omega_{11}}{\omega_{00}}\mathbf{f} + (\cdots)\mathbf{a} + (\cdots)\mathbf{b},$$

$$\mathbf{r}_{vv} = m(m-1)\frac{\omega_{02}}{\omega_{00}}\mathbf{d} + (\cdots)\mathbf{b}.$$

Also note that $\mathbf{n} = \mathbf{a} \times \mathbf{b}/\|\mathbf{a} \times \mathbf{b}\|$. By the definition (6) of L, M and N, we have

$$L = n(n-1)\frac{\omega_{20}}{\omega_{00}}h_{20}, \quad M = nm\frac{\omega_{11}}{\omega_{00}}h_{11}, \quad N = m(m-1)\frac{\omega_{02}}{\omega_{00}}h_{02}. \tag{10}$$

Now substituting (9) and (10) into (3) and (4), and doing some simplifications, we get the formulas for Gaussian and mean curvatures

$$K_g = \left(\frac{\omega_{00}}{\omega_{10}\omega_{01}}\right)^2 \frac{\frac{n-1}{n}\,\frac{m-1}{m}\omega_{20}\omega_{02}h_{20}h_{02} - \omega_{11}^2 h_{11}^2}{\|\mathbf{a} \times \mathbf{b}\|^2}, \tag{11}$$

$$K_m = \frac{\omega_{00}}{\omega_{10}^2\omega_{01}^2}\ \frac{\frac{n-1}{n}\omega_{01}^2\mathbf{b}^2\omega_{20}h_{20} - 2\omega_{10}\omega_{01}(\mathbf{a}\cdot\mathbf{b})\omega_{11}h_{11} + \frac{m-1}{m}\omega_{10}^2\mathbf{a}^2\omega_{02}h_{02}}{2\|\mathbf{a}\times\mathbf{b}\|^2}. \quad (12)$$

If we further introduce notations

$$\begin{aligned}
&\tilde{a} = \frac{\omega_{10}}{\omega_{00}}\|\mathbf{a}\|, \quad \tilde{b} = \frac{\omega_{01}}{\omega_{00}}\|\mathbf{b}\|, \qquad \tilde{S} = \frac{1}{2}\,\|\frac{\omega_{10}}{\omega_{00}}\mathbf{a}\times\frac{\omega_{01}}{\omega_{00}}\mathbf{b}\| = S\frac{\omega_{10}\omega_{01}}{\omega_{00}^2},\\
&\tilde{h}_{11} = \frac{\omega_{11}}{\omega_{00}}h_{11}, \ \ \tilde{h}_{20} = \frac{n-1}{n}\ \frac{\omega_{20}}{\omega_{00}}h_{20}, \ \ \tilde{h}_{02} = \frac{m-1}{m}\ \frac{\omega_{02}}{\omega_{00}}h_{02},
\end{aligned}$$

then the formulas can be symbolically simplified to

$$K_g = \left(\tilde{h}_{20}\,\tilde{h}_{02} - \tilde{h}_{11}^2\right)/4\tilde{S}^2, \tag{13}$$

$$K_m = \left(\tilde{h}_{20}\,\tilde{b}^2 - 2\tilde{h}_{11}\,\tilde{a}\,\tilde{b}\,\cos\theta + \tilde{h}_{02}\,\tilde{a}^2\right)/8\tilde{S}^2. \tag{14}$$

Remark 1. Obviously, formulas (11) and (12) or (13) and (14) just contain some simple geometric quantities, like (scaled) length or area, related to the control mesh of the rational Bézier patch. Compared to the formulas (3) and (4), they are more intuitive and also simpler to compute.

Remark 2. Though the equations derived above are valid at the bottom-left corner, by symmetry similar formulas are easily written out at the other three corners of the rational Bézier patch. Moreover, at any point of the surface other than the four corners, the formulas can also be used to calculate curvatures with help of subdividing the surface there.

Remark 3. According to differential geometry, from Gaussian and mean curvatures, we can compute the principal curvatures

$$k_{1,2} = K_m \pm \sqrt{K_m^2 - K_g}.$$

Meanwhile, the principal directions of curvatures can be determined by

$$\frac{du}{dv} = -\frac{M - k_{1,2}F}{L - k_{1,2}E} = -\frac{N - k_{1,2}G}{M - k_{1,2}F}.$$

3 Curvatures of triangular rational Bézier surfaces

A triangular rational Bézier patch of degree n is defined by

$$\mathbf{r}(u,v) = \frac{\sum\limits_{i+j+k=n} \omega_{ijk}\mathbf{P}_{ijk}B^n_{ijk}(u,v)}{\sum\limits_{i+j+k+n} \omega_{ijk}B^n_{ijk}(u,v)} \tag{15}$$

where $B^n_{ijk}(u,v) = \dfrac{n!}{i!j!k!}u^iv^j(1-u-v)^k$ are Bernstein polynomials; $\mathbf{P}_{ijk}$ are the control points, forming a triangular control mesh (see Figure 3); and ω_{ijk} are the weights.

As in the tensor-product case, we only consider the curvatures at corner $(u,v) = (0,0)$. Although we can follow the straightforward approach of Section 2, i.e., computing the partial derivatives of the triangular Bézier patch, here we take another approach. Since the triangular patch $\mathbf{r}(u,v)$ can be considered as a degree $n \times n$ tensor-product rational Bézier patch, we can utilize the established formulas (11) and (12).

Let $\mathbf{a} = \mathbf{P}_{10(n-1)} - \mathbf{P}_{00n}$, $\mathbf{b} = \mathbf{P}_{01(n-1)} - \mathbf{P}_{00n}$, S denote the area of the triangle $\mathbf{P}_{00n}\mathbf{P}_{10(n-1)}\mathbf{P}_{01(n-1)}$, θ be the angle between vectors $\mathbf{P}_{00n}\mathbf{P}_{10(n-1)}$ and $\mathbf{P}_{00n}\mathbf{P}_{01(n-1)}$, and h_{ij} be the signed distance from $\mathbf{P}_{ij(n-i-j)}$ to the plane spanned by $\mathbf{P}_{00n}, \mathbf{P}_{10(n-1)}$ and $\mathbf{P}_{01(n-1)}$. Suppose the tensor-product representation of the patch (15) is

$$\mathbf{r}(u,v) = \frac{\sum\limits_{i=0}^{n}\sum\limits_{j=0}^{n} \bar{\omega}_{ij}\bar{\mathbf{P}}_{ij}B^n_i(u)B^n_j(v)}{\sum\limits_{i=0}^{n}\sum\limits_{j=0}^{n} \bar{\omega}_{ij}B^n_i(u)B^n_j(v)}.$$

It is easy to show that $\bar{\mathbf{P}}_{ij} = \mathbf{P}_{ij(n-i-j)}$ and $\bar{\omega}_{ij} = \omega_{ij(n-i-j)}$ for $(i,j) = (0,0), (1,0)$, and $(0,1)$. Thus the geometric quantities, like $\mathbf{a}$, $\mathbf{b}$, h_{ij}, and etc., are the same as their counterparts in the tensor-product representation except for h_{11}. Therefore we only need to check $\bar{\mathbf{P}}_{11}$ and $\bar{\omega}_{11}$. Considering the mixed derivative of the denominators of the above two representations at $(u,v) = (0,0)$, we get

$$\bar{\omega}_{11} = \frac{n-1}{n}\omega_{11(n-2)} + \frac{\omega_{10(n-1)} + \omega_{01(n-1)} - \omega_{00n}}{n}$$

Similarly, by considering the numerators, we have

$$\bar{\mathbf{P}}_{11} = \frac{n-1}{n}\frac{\omega_{11(n-2)}}{\bar{\omega}_{11}}\mathbf{P}_{11(n-2)} + \frac{\omega_{10(n-1)}\mathbf{P}_{10(n-1)} + \omega_{01(n-1)}\mathbf{P}_{01(n-1)} - \omega_{00n}\mathbf{P}_{00n}}{n\bar{\omega}_{11}}$$

If we denote the signed distance of $\bar{\mathbf{P}}_{11}$ to the plane $\bar{\mathbf{P}}_{00}\bar{\mathbf{P}}_{10}\bar{\mathbf{P}}_{01}$ by $\bar{h}_{11}$, then $\bar{\omega}_{11}\bar{h}_{11} = \frac{n-1}{n}\omega_{11(n-2)}h_{11}$. We substitute all the above relations into formulas (11) and (12), and the formulas of Gaussian and mean curvatures for the triangular rational Bézier patch (15) are

$$K_g = \left(\frac{n-1}{n}\right)^2 \left(\frac{\omega_{00n}}{\omega_{10(n-1)}\omega_{01(n-1)}}\right)^2 \frac{\omega_{20(n-2)}\omega_{02(n-2)}h_{20}h_{02} - \omega_{11(n-2)}^2 h_{11}^2}{\|\mathbf{a}\times\mathbf{b}\|^2}, \tag{16}$$

$$K_m = \frac{n-1}{n} \frac{\omega_{00n}}{\omega_{10(n-1)}^2\omega_{01(n-1)}^2} \cdot \frac{\omega_{01(n-1)}^2\mathbf{b}^2\omega_{20(n-2)}h_{20} - 2\omega_{10(n-1)}\omega_{01(n-1)}(\mathbf{a}\cdot\mathbf{b})\omega_{11(n-2)}h_{11} + \omega_{10(n-1)}^2\mathbf{a}^2\omega_{02(n-2)}h_{02}}{2\|\mathbf{a}\times\mathbf{b}\|^2}, \tag{17}$$

or just (13) and (14) if the following notations are adopted

$$\tilde{a} = \frac{\omega_{10(n-1)}}{\omega_{00n}}\|\mathbf{a}\|, \qquad \tilde{b} = \frac{\omega_{01(n-1)}}{\omega_{00n}}\|\mathbf{b}\|,$$
$$\tilde{S} = S\frac{\omega_{10(n-1)}\omega_{01(n-1)}}{\omega_{00n}^2}, \qquad \tilde{h}_{ij} = \frac{n-1}{n}\frac{\omega_{ij(n-i-j)}}{\omega_{00n}}h_{ij}.$$

References

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