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# Implicit Representation of Parametric Curves and Surfaces*

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The following two problems are shown to have closed-form solutions requiring only the arithmetic operations of addition, subtraction, multiplication and division: (1) Given a curve or surface defined parametrically in terms of rational polynomials, find an implicit polynomial equation which defines the same curve or surface. (2) Given the Cartesian coordinates of a point on such a curve or surface, find the parameter(s) corresponding to that point. It is shown that a two-dimensional curve defined parametrically in terms of rational degree $n$ polynomials in $t$ can be expressed implicitly as a degree $n$ polynomial in $z$ and $y$. It is also demonstrated that a "bi-$m$-ic" parametric surface (where e.g., $m = 3$ for bicubic) can be expressed implicitly as a polynomial in $x$, $y$, $z$ of degree $2m^2$. The degree of a rational bi-$m$-ic surface is also shown to be $2m^2$. The application of these results to finding curve and surface intersections is discussed. © 1984 by Academic Press, Inc.

## 1. INTRODUCTION

This paper provides solutions to two problems which are fundamental to computer-aided geometric design: the implicitization and inversion of parametric polynomial curves and surfaces. Implicitization means the conversion of parametrically defined curves or surfaces into their implicit forms, and inversion is the process of finding the parameter(s) corresponding to a point on a parametrically defined curve or surface whose Cartesian coördinates are known.

These solutions demand the resurrection of the theory of elimination—a technique which was very popular in algebraic geometry early in this century, but was later abandoned in favor of less constructive methods, and in whose absence these two problems knew no closed-form solution [6, 15]. Since the general theory of elimination is a lost art, with obscure references, occasion is taken to explain it in some detail, with particular application to the mentioned problems above.

### 1.1. Notation

This paper will adhere to the following notational conventions. All functions are polynomials. A superscript on the function name indicates the degree of the polynomial. Hence, $F^n(x, y, z)$ represents an $n$th degree polynomial in $x$, $y$, and $z$.

Superscripts in a list of variables indicate the maximum degree to which the superscripted variable appears in the polynomial. Furthermore, a letter separated

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from the parameter list by a vertical bar indicates the coefficient symbol. For example, $f(a|s^n, t^m)$ signifies $\Sigma_{i=0}^{n}\Sigma_{j=0}^{m}a_{ij}s^it^j$.

## 2. IMPLICIT AND PARAMETRIC REPRESENTATION

An implicit representation of a surface is an equation of the form $f(x, y, z) = 0$ where $x$, $y$, $z$ are cartesian coordinates of points on the surface. A two-dimensional (2-D) curve has the implicit form $f(x, y) = 0$. Curves and surfaces which can be expressed implicitly in terms of a polynomial equations are known as algebraic curves and surfaces.

A rational parametric surface is defined: $x = x(s,t)/w(s,t)$, $y = y(s,t)/w(s,t)$, $z = z(s,t)/w(s,t)$. A rational parametric curve, 2-D or 3-D, is expressed as a function of a single parameter: $x = x(t)/w(t)$, $y = y(t)/w(t)$, and for 3-D, $z = z(t)/w(t)$. An integral parametric curve or surface is a rational parametric whose denominator $w$ equals one.

The parametric curves (and surfaces) discussed in this paper are assumed to be irreducible in the sense that a change of variables cannot reduce the degree of the polynomial expressions. This amounts to having a one-to-one correspondence between parameter values and points with the permissible exception of a finite number of multiple points (and/or multiple curves). The conditions under which the degree of a parametric equation can be reduced are addressed in Luroth's theorem [14].

### 2.1. *Why Implicitize*

It is universally recognized that the parametric representation is best suited for generating points along a curve or surface, whereas the implicit representation is most convenient for determining whether a given point lies on a specific curve or surface. This motivates the search for a means of converting from one representation to the other. Further motivation is provided by intersection problems encountered in surface and solid modeling systems. These problems are greatly simplified if one of the curves or surfaces can be expressed implicitly and the other parametrically. In such a case, the parametric expressions for one surface, $x(s,t)$, $y(s,t)$, $z(s,t)$, can be substituted directly into the implicit equation of the other surface $f(x, y, z) = 0$ to yield a single equation $f[x(s,t), y(s,t), z(s,t)] = 0$ which expresses the curve of intersection implicitly in parameter space.[1] Converting from parametric to implicit form is called implicitization; converting from implicit to parametric form is called parameterization or uniformization. We shall return to the uniformization problem in Section 7, for now we shall focus our attention on the problem of implicitization.

With such important motivation for implicitization the apparent absence of any such technique has been bemoaned [6]. For example, the literature is void of any description of what the implicit form of a bicubic patch would be, let alone any method for producing one. Indeed, in the absence of elimination theory, the implicitization problem appears to have no general solution.

We now proceed to solve the implicitization problem. Since the primary tool of elimination theory is not widely known, a brief tutorial is presented first.

## 3. FUNDAMENTALS OF ELIMINATION THEORY

Elimination theory investigates the conditions under which sets of polynomials have common roots. Usually, the theory concerns itself with sets of $n$ homogeneous

[1] In general, there is no rational polynomial for this intersection curve (see Sect. 7).

equations in $n$ variables, $F_1(x_1, \dots, x_n) = 0, \dots, F_n(x_1, \dots, x_n) = 0$. Two important special cases of such systems of equations are first, the case where the equations are all linear in the unknowns and second, the case $n = 2$.

It is instructive to consider first the special case where the polynomials are all linear, since here we can observe a familiar example of what has been called the fundamental theorem of elimination [8]:

THEOREM 1. *Given a system of n homogeneous linear equations in n unknowns,*

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \vdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix} = 0 \qquad or \qquad Ax = 0,$$

*the system can have a nontrivial solution (i.e., a solution other than* $x_1 = \cdots = x_n = 0$*) if and only if the determinant of the coefficient matrix vanishes (i.e.,* $|A| = 0$*).*

A related theorem will be used to solve the inversion problem:

THEOREM 2. *Given a set of* $n - 1$ *homogeneous linear equations in n unknowns, the ratio of any two unknowns* $x_i/x_j$ *can be found by taking the ratio of the determinants of the coefficient matrix with columns i and j, respectively, deleted and multiplying by* $(-1)^{i+j}$.

Proofs of these theorems can be found in most linear algebra texts.

Elimination theory is mainly concerned with finding a relationship between the coefficients of a set of polynomials, such as the relationship $|A| = 0$ in Theorem 1, which can be used to determine whether the polynomials have a common nontrivial solution.

DEFINITION. A resultant of a set of polynomials is an expression involving the coefficients of the polynomials such that the vanishing of the resultant is a necessary and sufficient condition for the set of polynomials to have a common nontrivial root.

We next examine the resultant of two polynomials in one variable. There are several ways of finding such a resultant; [13] lists half a dozen such methods. The method most commonly found in text books of the past fifty years is based on symmetric functions. This approach seems to be preferable from a theoretic point of view, but is not as practical as the methods which follow.

### *3.1. Sylvester's Dialytic Expansion*

This method of finding resultants invokes the device of considering all the individual monomials of a polynomial as independent variables. Hence, $x^2$ and $x$ are taken to be independent. A number of auxiliary equations are generated simply by multiplying the initial polynomials by well-chosen monomials such that the total number of equations equals the total number of monomials (or "independent variables").

This technique can be more readily understood from the following example:

EXAMPLE 1. Find the resultant of two quadratic polynomials in one variable using Sylvester's method.

Let the polynomial equations be written as $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$. Thus we initially have two homogeneous equations in three "independent variables", $x^2$, $x$, and 1. By multiplying both equations by $x$, we form four auxiliary equations with four "independent variables": $x^3, x^2, x, 1$. These equations can be written as

$$\begin{bmatrix} a & b & c & 0 \\ 0 & a & b & c \\ a' & b' & c' & 0 \\ 0 & a' & b' & c' \end{bmatrix} \begin{Bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{Bmatrix} = 0 \qquad \text{or} \qquad AX = 0.$$

We want the conditions under which the original equations have a common root. This is equivalent to asking for the conditions under which the above set of equations can be satisfied by some $x$. By Theorem 1, those necessary and sufficient conditions are expressed in the vanishing of $|A|$. Therefore, the resultant is $|A|$.

To illustrate, consider two examples. The resultant of $f(x) = x^2 - 6x + 2$ and $g(x) = x^2 + x + 5$ is 233, so $f$ and $g$ do not have a common root. The resultant of $f(x) = x^2 - 4x + 5$ and $g(x) = x^2 - 7x + 10$ is 0, indicating that they do have a common root. That common root is the number 5. Although the resultant does not directly reveal the common root, it can be found from the resultant matrix using Theorem 2 as follows: The first step is to arbitrarily discard any row of the resultant matrix. Discarding the fourth row leaves

$$\begin{bmatrix} 1 & -4 & -5 & 0 \\ 0 & 1 & -4 & -5 \\ 1 & -7 & 10 & 0 \end{bmatrix} \begin{Bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{Bmatrix} = 0.$$

Using Theorem 2, the common root $x$ can be found as either $x^3/x^2$, $x^2/x$ or $x/1$. For example,

$$x = \frac{x^3}{x^2} = -\begin{vmatrix} -4 & -5 & 0 \\ 1 & -4 & -5 \\ -7 & 10 & 0 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & -5 & 0 \\ 0 & -4 & -5 \\ 1 & 10 & 0 \end{vmatrix} = 5.$$

In the original equations, if $a = a' = 0$ or if $c = c' = 0$, $|A|$ vanishes identically. Some writers have suggested that the restriction must be made that either $a$ or $a'$ and either $c$ or $c'$ must be nonzero for the resultant to be valid. However, we note that in either case, the vanishing of the resultant does correctly identify the existence of a common root. In the first case, the common root is infinity and in the second case, the common root is zero.

In general, Sylvester's method can be used to express the resultant of two polynomials of degree $m$ and $n$, respectively, as a determinant of a matrix with $m + n$ rows and columns.

### *3.2. Cayley's Statement of Bezout's Method*

This alternate method for finding the resultant of two polynomials is presented for three reasons: first, because it produces a more compact expression for the resultant than Sylvester's method; second, because a similar technique will be used later to implicitize a surface; and third, because it generates all the necessary auxiliary polynomials in one pass without the need to find clever multiplying factors.

Cayley noted that if two polynomials $f(x)$ and $g(x)$ have a common root $x = x_0$ then the equation $f(x)g(\alpha) - f(\alpha)g(x) = 0$ will be satisfied by that common root for any value of $\alpha$. Since the equation will always be satisfied for $x = \alpha$ (even if there is no common root), the expression must contain $(x - \alpha)$ as a factor. After dividing by $(x - \alpha)$, are are left with a polynomial which we choose to view as having monomials in $1, \alpha, \alpha^2, \ldots,$ where the coefficient of each monomial is a polynomial in $x$. Since at the common root $x = x_0$ the entire expression must vanish for any value of $\alpha$, each of the coefficient polynomials in $x$ must vanish at $x_0$. We have therefore generated a set of polynomials similar to those obtained from Sylvester's method, but in a more compact form. The following example will illustrate:

EXAMPLE 2. Find the resultant of two nonhomogeneous quadratic polynomials using Cayley's method. Given $f(x) = a^2 + bx + c$ and $g(x) = a'x^2 + b'x + c'$, we set up Cayley's device as follows:

$$\frac{f(x)g(\alpha) - f(\alpha)g(x)}{(x-\alpha)}$$
$$= \frac{(ax^2 + bx + c)(a'\alpha^2 + b'\alpha + c') - (a\alpha^2 + b\alpha + c)(a'x^2 + b'x + c')}{(x-\alpha)}$$
$$= [(ab' - a'b)x + (ac' - a'c)]\alpha + [(ac' - a'c)x + (bc' - b'c)].$$

For a common root $x = x_0$ to exist, we must have

$$\begin{bmatrix} (ab' - a'b) & (ac' - a'c) \\ (ac' - a'c) & (bc' - b'c) \end{bmatrix} \begin{Bmatrix} x \\ 1 \end{Bmatrix} = 0 \qquad \text{or} \qquad Ax = 0.$$

From Theorem 1, this can only be true if $|A| = 0$, so the resultant is $|A|$. This is known as Bezout's form of the resultant.

### 3.3. *Resultant of Three Polynomials*

Once again, the general strategy is to generate a collection of polynomials equal in number to the number of different terms. The methods of Sylvester and Cayley can, with modification, find the resultant of three polynomials in two variables. These methods will be discussed in Section 5 where they are applied to surface implicitization.

## 4. IMPLICITIZING PLANAR CURVES

### 4.1. *Implicitization of Integral Parametric Curves*

It is a simple matter to implicitize a planar curve using elimination theory, and in fact the problem was addressed specifically in [13, 11, 10, and 12]. Implicitization of a parametric quadratic will illustrate the technique. Given $x = a_2t^2 + a_1t + a_0$ and $y = b_2t^2 + b_1t + b_0$ the equations are rewritten as $a_2t^2 + a_1t + (a_0 - x) = 0$ and $b_2t^2 + b_1t + (b_0 - y) = 0$. We now view $(a_0 - x)$ and $(b_1 - y)$ as the constant terms. Since the resultant expresses the relationship which must exist among the coefficients in order for there to exist a $t$ which simultaneously satisfies both equations, the resultant itself is the implicit form of the parametric curve.

Expanding the resultant obtained previously by Sylvester's method,

$$\begin{vmatrix} a_2 & a_1 & (a_0 - x) & 0 \\ 0 & a_2 & a_1 & (a_0 - x) \\ b_2 & b_1 & (b_0 - y) & 0 \\ 0 & b_2 & b_1 & (b_0 - y) \end{vmatrix}$$

yields

$$b_2^2 x^2 - 2a_2 b_2 xy + a_2^2 y^2 + \left(-2a_0 b_2^2 + a_1 b_1 b_2 - a_2 b_1^2 + 2a_2 b_0 b_2\right)x$$
$$+\left(-2b_0 a_2^2 + b_1 a_1 a_2 - b_2 a_1^2 + 2b_2 a_0 a_2\right)y + \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & 0 \\ 0 & b_2 & b_1 & b_0 \end{vmatrix} = 0.$$

It is noteworthy that the discriminant of this conic curve is identically zero, which demonstrates the well-known fact that a parametric quadratic curve is a parabola.

In general a curve defined parametrically by $x = f^n(t)$, $y = g^n(t)$ will have an implicit form $h^n(x, y) = 0$.

*4.2. Implicitization of Rational Parametric Curves*

There are a number of strategies one can employ to implicitize a rational parametric. We will examine two such approaches. The first is an adaptation of the method for integral parametrics and the second is based on homogeneous polynomials.

The first approach will be illustrated with rational quadratics. Given

$$x = \frac{a_2 t^2 + a_1 t + a_0}{d_2 t^2 + d_1 t + d_0}, \qquad y = \frac{b_2 t^2 + b_1 t + b_0}{d_2 t^2 + d_1 t + d_0},$$

we can write

$$(d_2 x - a_2)t^2 + (d_1 x - a_1)t + (d_0 x - a_0) = 0$$

and

$$(d_2 y - b_2)t^2 + (d_1 y - b_1)t + (d_0 y - b_0) = 0$$

and eliminate $t$ as before. This leads to a $4 \times 4$ determinant if we adopt Sylvester's method and a $2 \times 2$ determinant if we use Cayley's technique. In both cases, the implicit equation is a degree 2 polynomial in $x, y$.

In the second approach the equations are written homogeneously as

$$x = a_2 t^2 + a_1 tu + a_0 u^2$$
$$y = b_2 t^2 + b_1 tu + b_0 u^2$$
$$w = d_2 t^2 + d_1 tu + d_0 u^2.$$

Multiplying these equations by $u$ and $t$, respectively, generates six equations with six unknowns:

$$\begin{bmatrix} x & 0 & a_2 & a_0 & 0 & 0 \\ y & 0 & b_2 & b_1 & b_0 & 0 \\ w & 0 & d_2 & d_1 & d_0 & 0 \\ 0 & x & 0 & a_2 & a_1 & a_0 \\ 0 & y & 0 & b_2 & b_1 & b_0 \\ 0 & w & 0 & d_2 & d_1 & d_0 \end{bmatrix} \begin{Bmatrix} -t \\ -u \\ t^3 \\ ut^2 \\ u^2t \\ u^3 \end{Bmatrix} = 0.$$

This implicitization yields a second degree homogeneous polynomial in $x$, $y$, and $w$, but because we only care about the ratios of the homogeneous coordinates, we can simply set $w$ to 1, since this is equivalent to dividing by $w^2$.

From the second approach, it is evident that the implicit form of a 2-D curve defined parametrically by rational polynomials of degree $n$ will be a polynomial in $x$ and $y$ of degree $n$.

### *4.3. Inversion of Curves*

The inversion problem is solvable immediately using the resultant matrix. We illustrate with Sylvester's method. Consider the quadratic case $x = a_2t^2 + a_1t + a_0$, $y = b_2t^2 + b_1t + b_0$. The equations defining the resultant matrix are

$$\begin{bmatrix} a_2 & a_1 & (a_0 - x) & 0 \\ 0 & a_2 & a_1 & (a_0 - x) \\ b_2 & b_1 & (b_0 - y) & 0 \\ 0 & b_2 & b_1 & (b_0 - y) \end{bmatrix} \begin{Bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{Bmatrix} = 0.$$

Our goal is to find the value of $t$ corresponding to a point $(x, y)$ known to be on the curve. Since all four of the above equations are satisfied by the desired $t$, any row of the matrix may be deleted to leave three homogeneous equations in the four unknowns: $t^3$, $t^2$, $t$, 1. Using Theorem 2, the ratio of any two adjacent terms can be found and that ratio will equal $t$. In the event the point lies on a self-intersection, all of the unknowns vanish and an approach such as mentioned in [13] must be used.

This inversion technique demonstrates the interesting result that for a parametric curve whose defining polynomials have rational coefficients, each point on the curve with rational $(x, y)$ coordinates has a corresponding parameter value that is rational. This is known as birational correspondence.

## 5. IMPLICITIZING SURFACES

A general bivariate surface can be expressed parametrically as

$$P_z(s, t) = f1(a|s^m, t^n) - x = 0$$
$$P_y(s, t) = f2(b|s^m, t^n) - y = 0$$
$$P_z(s, t) = f3(c|s^m, t^n) - z = 0.$$

The implicit form of the surface equation is desired. It must be of the form $f(x, y, z) = 0$, and only $(x, y, z)$ points on the surface must satisfy the equation. Implicitization of a surface may therefore be viewed as determining the conditions under which the polynomials, $P_x, P_y, P_z$ can be satisfied simultaneously; therefore $f(x, y, z)$ is the resultant of the polynomials $P_x, P_y, P_z$.

The strategy here is analogous to that for implicitizing curves. Once again, our goal is to arrive at a number of auxiliary equations, deduced from the parametric surface equations, equal in number to the number of variables. Sylvester's dialytic method can be applied to the bilinear surface, but for bivariate surfaces of higher degree, that method can introduce linearly dependent equations which cause the resultant to vanish identically. An extension of Cayley's method will be used to generate a resultant matrix for a general bivariate surface.

Despite its limitations, Sylvester's method provides a better introduction to surface implicitization than does Cayley's method. Subsection 5.1 demonstrates bilinear surface implicitization using Sylvester's method.

### *5.1. Bilinear Surface Implicitization*

A bilinear surface has the form

$$P_x(s,t) = (a_0 - x) + a_1 s + a_2 t + a_3 st = 0$$
$$P_y(s,t) = (b_0 - y) + b_1 s + b_2 t + b_3 st = 0$$
$$P_z(s,t) = (c_0 - z) + c_1 s + c_2 t + c_3 st = 0.$$

These equations can be multiplied by either $s$ or $t$ to form six "independent variables." Multiplying by $s$, the following set of equations arise:

$$\begin{bmatrix} (a_0 - x) & a_1 & a_2 & a_3 & 0 & 0 \\ (b_0 - y) & b_1 & b_2 & b_3 & 0 & 0 \\ (c_0 - z) & c_1 & c_2 & c_3 & 0 & 0 \\ 0 & (a_0 - x) & 0 & a_2 & a_1 & a_3 \\ 0 & (b_0 - y) & 0 & b_2 & b_1 & b_3 \\ 0 & (c_0 - z) & 0 & c_2 & c_1 & c_3 \end{bmatrix} \begin{Bmatrix} 1 \\ s \\ t \\ st \\ s^2 \\ s^2 t \end{Bmatrix} = 0.$$

From the same reasoning we applied to curves, the implicit form of the bilinear surface is the determinant of the coefficient matrix. This equation is clearly second degree in $x, y, z$ and it is, in fact, the equation of a hyperbolic paraboloid.

### *5.2. Extension of Cayley's Method*

Dixon made the following observation [5]: If there exists an $(s', t')$ which will simultaneously satisfy $P_x = P_y = P_z = 0$, the following determinant will vanish for that value of $(s, t)$ regardless of the values of $\alpha$ and $\beta$.

$$\det(s, t, \alpha, \beta) = \begin{vmatrix} P_x(s,t) & P_y(s,t) & P_z(s,t) \\ P_x(\alpha,t) & P_y(\alpha,t) & P_z(\alpha,t) \\ P_x(\alpha,\beta) & P_y(\alpha,\beta) & P_z(\alpha,\beta) \end{vmatrix}.$$

This determinant vanishes for any $(s', t')$ which simultaneously satisfies $P_x = P_y = P_z = 0$ since the top row vanishes. Also, the determinant will vanish if either $s = \alpha$ or $t = \beta$ since then two rows would be identical. Hence, $(s - \alpha)$ and $(t - \beta)$ are factors of the determinant. Define

$$\delta(s, t, \alpha, \beta) = \frac{\det(s, t, \alpha, \beta)}{(s - \alpha)(t - \beta)}.$$

Clearly, $\delta$ will vanish for any value of $\alpha$ and $\beta$ if and only if $s = s'$ and $t = t'$. Note too that if the bivariate surface is of degree $n$ in $s$ and degree $m$ in $t$, then $\delta$ is of degree $n - 1$ in $s$, $2m - 1$ in $t$, $2n - 1$ in $\alpha$, and $m - 1$ in $\beta$. Now consider $\delta$ as a polynomial in $\alpha$ and $\beta$ whose coefficients are polynomials in $s$ and $t$:

$$\delta = \sum_{i=0}^{2n-1} \sum_{j=0}^{m-1} f_{i,j}(s, t)\alpha^i\beta^j.$$

This polynomial has $2mn$ terms. Since it must vanish for any value of $\alpha$ and $\beta$ if $s = s'$, $t = t'$, all of the $f_{i,j}(s', t')$ must vanish. Hence, $2mn$ polynomials have been generated, each of which has $2mn$ terms in $s$ and $t$ since $s$ appears in degree $n - 1$ and $t$ in degree $2m - 1$. The determinant of these coefficients will evidently serve as the resultant.

The set of equations can be expressed:

$$\begin{array}{l} \text{Polynomial} \\ \text{coefficients} \\ \alpha^0\beta^0 \quad : \\ \vdots \\ \alpha^i\beta^j \quad : \\ \vdots \\ \alpha^{2n-1}\beta^{m-1}: \end{array}
\begin{bmatrix} A(0,0,0,0) & \dots & A(0,0,k,l) & \dots & A(0,0,n-1,2m-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A(i,j,0,0) & \dots & A(i,j,k,l) & \dots & A(i,j,n-1,2m-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A(2n-1, m-1,0,0) & \dots & A(2n-1, m-1,k,l) & \dots & A(2n-1,m-1, n-1,2m-1) \end{bmatrix}
\begin{Bmatrix} s^0t^0 \\ \\ s^kt^l \\ \\ s^{n-1}t^{2m-1} \end{Bmatrix} = 0.$$

The desired resultants will be the determinant of the matrix.

The formula for computing each $A(i, j, k, l)$ will now be presented. The derivation of the formula is not straightforward and may be found in [5]. There it is shown that $A(i, j, k, l) = \Sigma(a_{IJ}, b_{KL}, c_{Mn})$ for all $I + K + M = i$ and $J + L' + N = l$, where $L = L' + j + 1$ and $M = M' + k + 1$ ($I, J, K, L', M', N$ are all nonnegative integers), and where $(a_{IJ}, b_{KL}, c_{MN})$ indicates the determinant of the $3 \times 3$ matrix whose principal diagonal is $a_{IJ}, b_{KL}, c_{MN}$. Note that only the first column of the above determinant can contain the variables $x, y, z$ because they only appear if both subscripts are zero, and $L$ and $M$ cannot be less than one. Therefore, all of the elements in the resultant matrix will be, at worst, linear in $x, y, z$ and the resultant will be of degree $2mn$ in $x, y, z$.

### 5.3. *Rational Bivariate Surfaces*

It will now be shown that the degree of a rational bivariate surface is also $2mn$. Let

$$\begin{aligned} D &= f(d|s^m, t^n), \\ x &= f(a|s^m, t^n)/D, \\ y &= f(b|s^m, t^n)/D, \\ z &= f(c|s^m, t^n)/D. \end{aligned}$$

The polynomials whose resultant is sought are

$$xD - f(a|s^m, t^n) = 0$$

$$yD - f(b|s^m, t^n) = 0$$

$$zD - f(c|s^m, t^n) = 0.$$

In this case, the coefficients of the resultant matrix are $A(i, j, k, l) = (d_{I,J}x - a_{I,J}, d_{K,L}y - b_{K,L}, d_{M,N}z - c_{M,N})$. Using the multilinearity of the determinant function, each $(d_{I,J}x - a_{I,J}, d_{K,L}y - b_{K,L}, d_{M,N}z - c_{M,N})$ can be expressed as the sum $(d_{IJ}x, d_{KL}y, d_{MN}z) + (d_{IJ}x, d_{KL}y, -c_{MN}) + \dots + (-a_{IJ}, -b_{KL}, -c_{MN})$. But any of these determinants which have more than one column involving $x$, $y$, or $z$ must vanish since any two such columns are linearly dependent. Hence, $(d_{IJ}x - a_{IJ}, d_{KL}y - b_{KL}, d_{MN}z - c_{MN}) = (d_{IJ}, -b_{KL}, -c_{MN})x + (-a_{IJ}, d_{KL}, -c_{MN})y + (-a_{IJ}, -b_{KL}, d_{MN})z + (-a_{IJ}, -b_{KL}, -c_{MN})$ which is linear in $x$, $y$ and $z$. Therefore, the implicit form of the rational bivariate surface is of degree $2mn$.

### 5.4. *Inversion of Surfaces*

As discussed, the resultant for implicitizing surfaces is the determinant of a matrix of coefficients for a set of $2mn$ polynomials whose terms are $s^i t^j$. The ratio of any of these terms can be found by Theorem 2, yielding $s = s^i t^j / s^{i-1} t^j$, $t = s^i t^j / s^i t^{j-1}$ for any available $i$ and $j$. The same discussion of self-intersections applies here as in Subsection 4.2.

## 6. APPLICATIONS TO CUBIC CURVES AND BICUBIC PATCHES

We now turn our attention to cubic curves and bicubic patches—the two most prominent types of curves and surfaces in computer aided geometric design.

The techniques discussed in Section 4 now make it not only possible but easy to find the implicit form of a parametric cubic curve. Since it is a degree 3 polynomial in $x$ and $y$, we can find all the intersection points of two planar parametric cubic curves simply by solving one degree 9 polynomial in $t$. This simple analytic method for finding all the intersection points is a major improvement over standard iterative techniques which only find one intersection point at a time and to do even that requires a good initial guess.

Similarly, the inversion problem can be solved directly by the given analytic method. This is a clear improvement over existing techniques which are based on iterative numerical approximation.

As for the bicubic patch, it can now be seen that the general bicubic patch is an algebraic surface of degree 18. This agrees with Kajiya's observation that a ray can intersect a bicubic patch in at most 18 points [7]. An algebraic surface $f^n(x, y, z) = 0$ has $(n + 1)(n + 2)(n + 3)/6$ terms, so the implicit equation of the bicubic patch has 1330 different terms! Using the method described above, it is possible to find the coefficients of these terms simply by expanding determinants.

Without elimination theory, the only solution to the inversion problem is to simultaneously solve a pair of degree 6 polynomials in two variables. The method presented in this paper solves the inversion problem in closed form, linear fashion. Specifically, the solution amounts to solving a set of 17 linear equations.

The nature of curves of intersection is also illuminated by these results. Because two surfaces of degree $m$ and $n$, respectively, intersect in a curve of degree $mn$, a bicubic patch in general intersects a plane in a curve of degree 18, a quadric surface in a curve of degree 36, a torus in a curve of degree 72, and another bicubic patch in a curve of degree 324. It should be realized that the intersection curve of degree $mn$ can be a collection of several distinct curves whose respective degrees sum to $mn$.

## 7. IMPLICIT TO PARAMETRIC CONVERSION

Having solved the implicitization problem, it is natural to wonder about going the other way; that is, can we always find a rational polynomial parameterization for an algebraic curve or surface. In general, we shall see that the answer is no, but for second-degree curves and surfaces this parameterization problem can always be solved. Indeed it is well known that for degree 2 all the cross terms can be removed by rotations. A simple but clever factorization technique can then be applied to parameterize the curve or surface. We shall illustrate the factorization technique for the circle and the sphere, for example, the circle: $x^2 + y^2 = R^2$. First we factor into linear factors

$$x \cdot x = (R + y)(R - y).$$

Rearranging these terms into rational linear factors and introducing the parameter $t$, we get

$$t = \frac{z}{R + y} = \frac{R - y}{z} \qquad \text{(Factorization step)}.$$

From the factorization step, we can generate two linear equations in $x, y$ with coefficients in $t$, and we can solve these equations for $x, y$ in terms of $t$. Thus $x - ty = Rt$ and $tx + y = R$, from which it is easily found that

$$x = \frac{2Rt}{1 + t^2} \qquad \text{and} \qquad y = \frac{R(1 - t^2)}{1 + t^2}.$$

To factor the sphere: $x^2 + y^2 + z^2 = R^2$, first we introduce an auxiliary variable $w$:

(1) $x^2 + y^2 = w^2$

(2) $w^2 + z^2 = R^2$.

Next using the technique of the preceding example, we separately parameterize each of these two equations. This leads to

$$x = \frac{2ws}{1 + s^2}, \qquad y = \frac{w(1 - s^2)}{1 + s^2}, \qquad w = \frac{2Rt}{1 + t^2}, \qquad z = \frac{R(1 - t^2)}{(1 + t^2)}.$$

Combining these results, we obtain

$$x = \frac{4Rst}{(1 + s^2)(1 + t^2)}, \qquad y = \frac{2R(1 - s^2)t}{(1 + s^2)(1 + t^2)}, \qquad z = \frac{R(1 - t^2)}{(1 + t^2)}.$$

Unfortunately, this factorization technique does not generalize to higher order curves and surfaces since even when we can still factor them, the equations we generate are no longer linear in $x$, $y$.

The problem of finding specific parametric representations for algebraic curves and surfaces is known classically as the uniformization problem. In 1865, Clebsch [3] proved the following uniformization theorem:

CLEBACH UNIFORMIZATION THEOREM. *An algebraic curve has a parametric rational polynomial representation if and only if the curve has genus zero.*

For an algebraic curve of degree $n$ with ordinary multiple points, the genus is the nonnegative integer defined by the formula

$$\text{genus} = \frac{(n-1)(n-2)}{2} - \Sigma \frac{r_i(r_i - 1)}{2},$$

where $r_i$ is the multiplicity of the $i$th multiple point. For further details, see [16]. It follows immediately from Clebsch's Uniformization Theorem that all degree 2 curves have parametric rational polynomial representations, and that there are degree 3 curves which fail to have such representations. This result explains why the factorization method works so nicely for conic sections but fails to generalize even to third degree curves. Similar results hold for surfaces.

Algebraic curves and surfaces of genus $> 0$ can still be parameterized, but they require more complicated functions. Clebsch [4] and Brill [2] showed that for curves of genus one and two, square roots are required. For curves of genus $> 2$ even more complicated functions are needed [9].

## 8. CONCLUSION

The study of curves and surfaces in computer-aided geometric design has not fully benefitted from the light cast on the subject by the great analytic geometers of the last century. This paper presents two important examples of problems deemed unsolvable in the CAD literature, which are, in fact, solvable using century-old theorems.

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