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Genus of the intersection curve of two rational surface patches

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Abstract. It is shown that two generic triangular surface patches with no base points and of parametric degree m and n respectively, intersect in a curve of degree m^2n^2 which is generally of genus $2m^2n^2 - \frac{3}{2}m^2n - \frac{3}{2}n^2m + 1$. Similarly, two generic tensor product surface patches of parametric degree $m_1 \times m_2$ and $n_1 \times n_2$ respectively, intersect in a curve of degree $4m_1m_2n_1n_2$ and generally of genus $8m_1m_2n_1n_2 - 2m_1m_2(n_1 + n_2) - 2n_1n_2(m_1 + m_2) + 1$. For example, two general bicubic patches in general position intersect in a curve of degree 324 and of genus 433. The significance of this genus value lies in the fact that only curves of genus 0 can be expressed parametrically using rational polynomials.

Genus and degree equations are also derived for intersection curves involving surface patches with simple base points. A class of surfaces is identified for which any plane section is a rational curve.

1. Introduction

The problem of computing the curve of intersection of two surfaces is a fundamental one in computer aided geometric design. It arises in a broad spectrum of tasks, ranging from constructive solid geometry [Requicha '83] to contouring of scattered data [Sabin '85].

It is well known in the CAGD community that the degree of the curve of intersection of two parametric surface patches is much larger than the degree of the parametric surface equations. For example, two generic bicubic patches intersect in a curve of degree 324. What is not widely appreciated in the CAGD community is that intersection curves *cannot* be represented exactly using parametric equations, even of degree 324.

The question of whether or not a given algebraic curve can be expressed using parametric equations can be answered by computing its *genus*. Algebraic geometers have known for over a century that only curves of genus 0 possess a parametric representation in terms of rational polynomials [Salmon '79, p.30], [Walker '50, p.67]. Curves of genus 1 [Abhyankar et al. '86] and 2 can be expressed parametrically using functions which involve the square root of a polynomial. General curves of genus greater than 2 have no rational parametrization, as well as no parametrization involving square roots [Hartshorne '77, p.347]. Of course, any algebraic curve can be expressed using power series, but only within a finite radius of convergence [de Montaudouin '86].

In this paper, we derive equations for the genus of the curve of intersection of two surfaces, and we show that in general the genus is quite large (the important point is that it is generally greater than two). For example, two general bicubic patches in general position intersect in a curve whose genus is 433. We also observe a family of surfaces for which any plane section curve is genus zero.

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Section 2 presents some necessary background on the genus equation for plane algebraic curves, on base points and intersections of algebraic surfaces. Section 3 discusses the singular loci of rational surfaces. Section 4 derives the genus equation, and section 5 presents a class of surfaces for which every plane section is a rational curve.

2. Background

Any point on a plane algebraic curve $f(x, y) = 0$ for which all zero through rth partial derivatives vanish is referred to as a *point of multiplicity* $r + 1$. Multiple points are also called singular points.

The *genus* of a plane algebraic curve $f(x, y) = 0$ of degree n is a function of the degree of the curve and the number and multiplicity of its singular points:

$$g = \tfrac{1}{2}(n-1)(n-2) - \tfrac{1}{2} \sum_{i=1}^{q} r_i(r_i - 1) \tag{1}$$

where g is the genus, n the degree, r_i the multiplicity of each singular point, and q is the number of singular points, provided that the singular points are 'ordinary', that is, near the ith singular point, the curve has exactly r_i complex branches. It should be emphasized that even though we are focusing on ordinary singularities, genus is also well defined for algebraic curves whose singularities are not ordinary [Abhyankar et al '86].

Curves whose genus is zero are called *rational* curves. Since only curves of genus zero can be expressed using rational polynomial parametric equations, Bézier curves and B-spline curves are by definition rational. It is noted from equation (1) that all degree two curves are rational, cubic curves with one double point are rational, quartic curves with three double points or one triple point are rational, etc.

We next examine the algebraic degree of a rational surface given in terms of parametric equations

$$x = \frac{f_1(s, t)}{f_4(s, t)}, \qquad y = \frac{f_2(s, t)}{f_4(s, t)}, \qquad z = \frac{f_3(s, t)}{f_4(s, t)}$$

(the f_i are all polynomials). The degree of such a surface can be determined by counting the number of times that it is intersected by a generic straight line. Consider two distinct planes in general position $a_1x + a_2y + a_3z + a_4 = 0$ and $b_1x + b_2y + b_3z + b_4 = 0$. These planes intersect the surface in curves

$$a_1f_1(s, t) + a_2f_2(s, t) + a_3f_3(s, t) + a_4f_4(s, t) = 0 \quad \text{and}$$
$$b_1f_1(s, t) + b_2f_2(s, t) + b_3f_3(s, t) + b_4f_4(s, t) = 0.$$

These curves are each degree n in s and t, where n is the largest of the degrees of the f_i. By Bezout's theorem, these two curves intersect in n^2 points, which must also be the number of times that the straight line common to the two planes intersects the surface. Thus, the degree of the surface, and of its implicit equation, is n^2.

The degree of the surface is diminished if there exists a *base point*, which is a parameter pair (s_b, t_b) for which $f_1(s_b, t_b) = f_2(s_b, t_b) = f_3(s_b, t_b) = f_4(s_b, t_b) = 0$. In this case, the two general plane section curves

$$a_1f_1(s, t) + a_2f_2(s, t) + a_3f_3(s, t) + a_4f_4(s, t) = 0 \quad \text{and}$$
$$b_1f_1(s, t) + b_2f_2(s, t) + b_3f_3(s, t) + b_4f_4(s, t) = 0$$

each contain the base point: they intersect at the base point and at $n^2 - 1$ other points. However, since the base point does not map to a unique point on the surface ($x = y = z = 0/0$

is undefined), this does not represent a point at which the straight line intersects the surface, and the degree of the surface is therefore n^2-1. Each additional simple base point diminishes the degree of the surface by one. Base points at infinity occur when all plane section curves have a common asymptotic direction.

To understand the influence of more complicated base points on the degree of the surface, consider the *linear system of plane curves*

$$a_1 f_1(s, t) + a_2 f_2(s, t) + a_3 f_3(s, t) + a_4 f_4(s, t) = 0.$$

Note that each curve in this linear system is the intersection of the surface with the plane $a_1 x + a_2 y + a_3 z + a_4 = 0$. A base point is clearly any point in common with all members of the linear system. If two general curves in the linear system are tangent at a base point, they intersect twice at the base point and the degree of the surface becomes n^2-2. If two general curves in the linear sytem have a double point in common, then they intersect four times at that base point and the degree becomes n^2-4. Thus, a general degree formula is $n^2-\rho$ where ρ is the total number of times that two general curves in the linear system intersect at base points. This also assumes that the surface has a one-to-one parametrization.

3. Singular loci

Note from equation (1) that for a properly parametrized rational curve of degree n,

$$\tfrac{1}{2}\sum_{i=1}^{q} r_i(r_i-1) = \tfrac{1}{2}(n-1)(n-2).$$

Generally, this means that there exist $\frac{1}{2}(n-1)(n-2)$ double points. This is the most double points that an irreducible degree n curve can possess.

Rational surfaces also contain singular loci, normally in the form of a double curve, or a curve of self intersection. When a plane intersects a surface containing a double curve, the plane section curve has a double point where the plane hits the double curve. This section derives an equation for the degree of that double curve. A good introduction to this topic is [Salmon '15, pp. 265–267]. More modern (and less accessible to someone not trained in algebraic geometry) references include [Kleiman '81] and [Piene '78].

We first consider the degree of the double curve on a triangular surface patch with no base points. By degree is meant the number of times the double curve intersects a general plane. If the surface is given by

$$x = \frac{f_1(s, t)}{f_4(s, t)}, \qquad y = \frac{f_2(s, t)}{f_4(s, t)}, \qquad z = \frac{f_3(s, t)}{f_4(s, t)},$$

where n is the largest degree of the f_i, then a plane section curve $a_1 x + a_2 y + a_3 z + a_4 = 0$ is a degree n curve in parameter space:

$$\beta(s, t) = a_1 f_1(s, t) + a_2 f_2(s, t) + a_3 f_3(s, t) + a_4 f_4(s, t) = 0.$$

A double point on $\beta(s, t) = 0$ can only occur if the surface has a double base point, or if the plane is a tangent plane. A general plane is not tangent to the surface, so $\beta(s, t) = 0$ has no double points and its genus is $\frac{1}{2}(n-1)(n-2)$.

Genus is invariant under birational transformations [Walker '50], so the plane section curve in three space must also have genus of $\frac{1}{2}(n-1)(n-2)$. Since the surface is degree $n^2-\rho$ (ρ defined as in the preceding section), the plane section is also degree $n^2-\rho$(by Bezout's theorem). However, a planar curve of degree $n^2-\rho$ and genus $\frac{1}{2}(n-1)(n-2)$ must have $\frac{1}{2}(n^2-\rho-1)(n^2-\rho-2)-\frac{1}{2}(n-1)(n-2)$ double points (from equation (1)). Double points on this plane section curve in $\mathbf{C}^3$ arise either at points where the plane is tangent to the surface, or at points where the plane cuts a double curve on the surface. Since a general plane is not tangent to the surface, all of the $\frac{1}{2}(n^2-\rho-1)(n^2-\rho-2)-\frac{1}{2}(n-1)(n-2)$ double points must be due to intersectins of the plane with a double curve. From Bezout's theorem, we deduce that the degree δ of the double curve equals the number of intersections it makes with the plane: δ of the double curve becomes

$$\delta=\tfrac{1}{2}(n^2-\rho-1)(n^2-\rho-2)-\tfrac{1}{2}(n-1)(n-2). \tag{2}$$

Equation (2) is only valid for surfaces with simple base points. In the event of multiple base points, a general plane section curve has multiple points and its genus therefore diminishes. This influences the computation of δ in a complicated way, a discussion of which is beyond the scope of this paper.

This degree formula can be illustrated by the case $n=2$, which is the Steiner surface [Sederberg et al. '85]. If there are no base points, a Steiner surface has a double curve of degree three. Actually, the double curve reduces to three double lines. If there is one base point, the double curve is degree one. In fact, a Steiner surface with one base point is a ruled cubic surface with a double line. If there are two base points, equation (2) indicates a double curve of degree 0, or no double curve at all. A Steiner surface with two base points degenerates to a quadric surface.

4. Genus of intersection curves

We now consider the genus of the curve of intersection of two triangular surface patches. For simplicity, we only consider surfaces with simple base points. The first surface has parametric equations of degree m, with ρ base points. The second surface has degree n parametric equations, and σ base points.

The first surface has an implicit equation of degree $m^2-\rho$. The intersection curve between the two surfaces can be expressed by substituting the parametric equations for surface two into the implicit equation of surface one. This yields a curve of degree $n(m^2-\rho)$. This curve has points of multiplicity $m^2-\rho$ at each of the σ base points. Also, since the double curve on the first surface is degree $\delta=\frac{1}{2}(m^2-\rho-1)(m^2-\rho-2)-\frac{1}{2}(m-1)(m-2)$, the second surface intersects that double curve in $\delta(n^2-\sigma)$ points, which appear as double points on the intersection curve.

In summary, the intersection curve is degree $n(m^2-\rho)$ in the parameter space of the second surface, it has σ points of multiplicity $(m^2-\rho)$, and it has $\delta(n^2-\sigma)$ double points. Thus, the *genus of the intersection curve of two triangular surface patches* is

$$\begin{aligned} g=&\tfrac{1}{2}\left[n(m^2-\rho)-1\right]\left[n(m^2-\rho)-2\right]-\tfrac{1}{2}\sigma(m^2-\rho)(m^2-\rho-1)\\ &-(n^2-\sigma)\left[\tfrac{1}{2}(m^2-\rho-1)(m^2-\rho-2)-\tfrac{1}{2}(m-1)(m-2)\right] \end{aligned} \tag{3}$$

which simplifies to

$$g=\tfrac{1}{2}\left[3(m^2-\rho)(n^2-\sigma)+m^2n^2-3(m^2-\rho)n-3(n^2-\sigma)m-\sigma\rho+2\right].$$

From a similar derivation, we can compute the *genus of the intersection of two tensor product surface patches* with simple base points. If the first patch has parametric degree $m_1\times m_2$ with ρ simple base points, and the second patch has parametric degree $n_1\times n_2$ with σ simple base points, then the genus of the general intersection curve works out to be

$$\begin{aligned} g=&8m_1m_2n_1n_2-2m_1m_2(n_1+n_2)-2n_1n_2(m_1+m_2)-\rho(n_1+n_2)\\ &-\sigma(m_1+m_2)+2\rho\sigma+3m_1m_2\sigma+3n_1n_2\sigma+1 \end{aligned} \tag{4}$$

Thus, the general intersection curve of two general bicubic patches, with no base points, is genus 433.

It can likewise be shown that the general curve of intersection of a *triangular surface patch* with degree m parametric equations and ρ simple base points and an *implicit surface* of degree n with no singular curve has degree $(m^3 - \rho)n$ and genus

$$g = \tfrac{1}{2}(mn-1)(mn-2) - \tfrac{1}{2}\rho n(n-1).$$

A degree $m_1 \times m_2$ *tensor product surface* with ρ simple base points generally intersects that same implicit surface in a curve of degree $(2m_1m_2 - \rho)n$ and genus

$$g = (m_1n-1)(m_2n-1) - \tfrac{1}{2}\rho n(n-1).$$

Finally, it is known classically [Semple & Roth '49, p. 90] that two general implicit surfaces of degrees m and n respectively, and with no double curve, generally intersect in a curve of degree mn and genus

$$g = \tfrac{1}{2}mn(m+n-4) + 1.$$

We emphasize that all of the preceding genus formula apply to the *general* intersection curve. If the two surfaces are positioned such that they are tangent at a point, then a double point occurs on the intersection curve at that point and the genus is one less than in the general case. For example, two quadrics generally intersect in a curve of degree four and of genus one. However, if those quadrics are tangent at one point, then the intersection curve is genus zero. Furthermore, if two quadrics have two points of tangency, then the intersection curve reduces to two curves of degree two. Point tangencies also occur when C^0 continuity is forced on two surface patches, and the intersection curve is reducible. For two bicubic patches, this means that the two surfaces intersect in the common boundary curve and generally in a residual curve of degree 321.

Tangencies do not occur in general, and intersection curves are generally irreducible.

5. Surfaces with rational plane section curves

It is well known that Steiner surface patches possess the nice property that any plane section is a rational curve [Salmon '12], [Sederberg et al. '85]. It is noteworthy that other surfaces, of higher degree, have this property as well.

If a parametric surface has a double base point, then any plane section has a double point in parameter space at that parameter pair. Thus, if a cubic triangular surface patch has a double base point, then any plane section curve is degree three in parameter space with a double point, making it a rational curve. Furthermore, any triangular surface patch with degree $m > 2$ parametric equations will have rational plane section curves if it has a base point of multiplicity $m-1$. Such a surface has an implicit equation of degree $2m-1$. A rational triangular Bézier surface of degree m has $\tfrac{1}{2}(m+1)(m+2)$ control points.

The most effortless way to impose an $\{m-1\}$-fold base point on a degree m triangular Bézier surface patch is to choose the base point to occur at $(s, t) = (0, 0)$. This is accomplished by assigning weights of zero to all the control points except the two rows of control points farthest from $\mathbf{P}_{00}$. Clearly, this leaves $2m+1$ control points to be freely chosen. It is not usually desirable to have the base point lie in the patch domain. In general, to impose a base point of multiplicity $m-1$, *any* $2m+1$ control points and their weights can be chosen, along with the parameter values at the base point, and the remaining control points and weights can be solved using linear equations.

It is shown in [Moore '87] that such surfaces are always ruled, except for the Steiner surface. It is not clear if these surfaces have a practical use, such as in contouring applications.

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