Isolator Polynomials

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Abstract

This paper explores the problem of isolating the real roots of a polynomial $p(x)$ with real coefficients, that is, of locating intervals which contain exactly one real root of p. A new solution to this problem is presented, consisting of finding a pair of auxiliary polynomials $a(x)$ and $b(x)$ whose set of combined real roots contain at least one value in every closed interval defined by each pair of adjacent real roots in p. It is shown that any member of the polynomial remainder sequence generated by p and p' can serve as one of these auxiliary polynomials. It is also shown that there exist $a(x)$ and $b(x)$ such that Degree(a) + Degree(b) = Degree(p)-1.

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1 Introduction

Polynomial real root isolation is a classical problem with an extensive literature (see, for example, [Collins and Akritas '76] and [Collins and Loos '76]). This paper presents a new approach to the root isolation problem by introducing, in section 2, the notion of isolator polynomials. Section 3 provides examples of isolator polynomials for low degree cases and Section 4 shows how polynomial remainders can be used as isolator polynomials.

2 Isolator Polynomials

We begin by proving
Theorem 1. Given any polynomial $p(x)$ with two adjacent real roots ρ_1 and ρ_2, and given any two other polynomials $b(x)$ and $c(x)$, define

$$a(x) = b(x)p'(x) + c(x)p(x). \tag{1}$$

Then $a(x)$ or $b(x)$ has at least one real root in the closed interval $[\rho_1, \rho_2]$.

Proof: Since $p(\rho_1) = 0$ and $p(\rho_2) = 0$,

$$\begin{aligned} a(\rho_1)a(\rho_2) &= b(\rho_1)p'(\rho_1)b(\rho_2)p'(\rho_2) \\ [a(\rho_1)a(\rho_2)] &= [b(\rho_1)b(\rho_2)]\,[p'(\rho_1)p'(\rho_2)] \end{aligned}$$

If ρ_1 and ρ_2 are distinct adjacent roots, then $p'(\rho_1)p'(\rho_2) < 0$. Thus, either $[a(\rho_1)a(\rho_2)] \leq 0$ or $[b(\rho_1)b(\rho_2] \leq 0$.
Q.E.D.

We refer to $a(x)$ and $b(x)$ as isolator polynomials, or *IP*s. Note that $b(x)$ and $c(x)$ can be chosen freely. As shall be shown, it is always possible to find a pair of IPs whose degrees sum to $degree(p) - 1$.

Note that Theorem 1 addresses distinct real roots. However, in the case of multiple roots, $p(x) = p'(x) = a(x) = 0$, and so $a(x)$ can be said to "isolate" a multiple root of $p(x)$ in the sense that a root of $a(x)$ occurs at exactly the same value as any multiple root of $p(x)$.

Section three works out in closed form some useful IPs for $p(x)$ of degree three, four, and five. Section four demonstrates that any member of the Sturm's sequence of p can serve as the a IP.

3 Motivating Examples

3.1 Degree Three Case.

For a cubic polynomial

$$p(x) = x^3 + p_2x^2 + p_1x + p_0 \tag{2}$$

we seek two linear isolator polynomials. Letting $b(x) = \frac{x}{3} + \frac{p_2}{9}$ and $c(x) = -1$, we have

$$a(x) = b(x)p'(x) + c(x)p(x) = \frac{2}{3}(\frac{1}{3}p_2^2 - p_1)x + (\frac{1}{9}p_2p_1 - p_0) \tag{3}$$

From Theorem 1, the roots of $a(x)$ and $b(x)$ isolate the roots of $p(x)$. The isolating values are

$$x_1 = -\frac{p_2}{3}; \quad x_2 = \frac{9p_0 - p_1p_2}{2p_2^2 - 6p_1}.$$

Note that from Vieta's formulas [Kurosh '75, p. 154], p_{n-1} is the negative of the sum of the roots of a degree n polynomial with $p_n = 1$. Thus, x_1 is the arithmetic mean of the three roots of this cubic p. If p has three real roots, it is obvious that the mean of those roots would lie between two of them. Furthermore, if the three roots are evenly spaced, then $x_1 = x_2 =$ the middle root.

Example: To illustrate, consider the cubic polynomial with roots $1, 2, 5$:

$$p(x) = (x - 1)(x - 2)(x - 5) = x^3 - 8x^2 + 17x - 10.$$

In this case, the isolating polynomials are

$$a(x) = \frac{26}{9}x - \frac{46}{9}; \quad b(x) = \frac{1}{3}x - \frac{8}{9}$$

whose roots are 1.769 and 2.667.

Roots of $p(x)$:	1		2		5
Roots of $a(x)$ and $b(x)$:		1.769		2.667	

3.2 Degree Four Case

The closed form expressions for the isolator polynomials are simplified if the polynomial is written in so-called general form:

$$p(x) = x^4 + p_2x^2 + p_1x + p_0.$$

The following linear $b(x)$ and quadratic $g(x)$ satisfy the requirements for isolator polynomials given in equation 1:

$$b(x) = -x,$$

$$a(x) = 2p_2x^2 + 3p_1x + 4p_0.$$

Once again, it turns out that the root of $b(x)$ (namely, $x = 0$), is the arithmetic mean of the four roots of p.

Example: Consider the polynomial with roots $-7, 1, 2, 4$:

$$p(x) = (x + 7)(x - 1)(x - 2)(x - 4) = x^4 - 35x^2 + 90x - 56.$$

(The coefficient of x^3 vanishes because we chose our roots to sum to zero.) In this case, $a(x) = -70x^2 + 270x - 224$ with roots 1.208 and 2.649. $b(x) = -x$ with root 0.

Roots of $p(x)$:	-7		1		2		4
Roots of $a(x)$ and $b(x)$:		0		1.208		2.649	

3.3 Degree Five Case

Once again, it simplifies matters if the polynomial is placed in general form:

$$p(x) = x^5 + p_3x^3 + p_2x^2 + p_1x + p_0.$$

The reader can verify that the quadratic $a(x)$ and quadratic $b(x)$

$$a(x) = (12p_3^3 + 45p_2^2 - 40p_3p_1)x^2 + (8p_3^2p_2 + 60p_2p_1 - 50p_3p_0)x + (4p_3^2p_1 + 75p_2p_0)$$

$$b(x) = 10p_3x^2 - 15p_2x + 4p_3^2$$

satisfy the conditions in equation 1 for isolator polynomials.

Example: Consider the polynomial with roots $-4, -3, 1, 2, 4$:

$$p(x) = (x+4)(x+3)(x-1)(x-2)(x-4) = x^5 - 23x^3 + 6x^2 + 112x - 96.$$

(The coefficient of x^4 vanishes because we chose our roots to sum to zero.) In this case, $a(x) = -41344x^2 - 44688x + 193792$ with roots -2.772 and 1.691. $b(x) = -230x^2 - 90x + 2116$ with roots -3.235 and 2.844.

Roots of $p(x)$:	-4		-3		1		2		4
Roots of $a(x)$ and $b(x)$:		-3.235		-2.772		1.691		2.844	

4 Polynomial Remainder Sequences

The examples in section 3 presented pairs of IPs whose degrees summed to $degree(p) - 1$. It turns out that such pairs can be generated from the polynomial remainder sequence of p and p' as follows:

$$\begin{aligned}
r_{-1}(x) &= p(x) \\
r_0(x) &= p'(x) \\
r_1(x) &= r_{-1}(x) - q_1(x)r_0(x) \\
r_2(x) &= r_0(x) - q_2(x)r_1(x) \\
r_3(x) &= r_1(x) - q_3(x)r_2(x) \\
&\cdot \\
&\cdot \\
&\cdot \\
r_i(x) &= r_{i-2}(x) - q_i(x)r_{i-1}(x)
\end{aligned}$$

Note that any member r_1 of the remainder sequence fits the recipe for the IP $a(x)$:

$$a(x) = b(x)p'(x) + c(x)p(x). \tag{4}$$

For example, in the equation

$$r_1(x) = p(x) - q_1(x)p'(x), \tag{5}$$

$a(x) = r_1(x)$, $b(x) = -q_1(x)$, and $c(x) = 1$. Likewise, $r_2(x)$ can also be made to fit the requirements for the isolator polynomial $a(x)$:

$$\begin{aligned}
r_2(x) &= p'(x) - q_2(x)r_1(x) \\
r_2(x) &= p'(x) - q_2(x)[p(x) - q_1(x)p'(x)] \\
r_2(x) &= [1 + q_2(x)q_1(x)]p'(x) - q_2(x)p(x)
\end{aligned}$$

In this case, $b(x) = 1 + q_2(x)q_1(x)$.

In general, we have

$$\begin{aligned} a_i(x) &= r_i(x) \\ b_i(x) &= b_{i-2}(x) - q_i(x)b_{i-1}(x) \end{aligned}$$

with $degree(a_i) \leq degree(p) - i - 1$ and $degree(a_i) + degree(b_i) = degree(p) - 1$.

Figure 1 shows *Mathematica* [Wolfram '88] code for computing a_i and b_i. A sample degree eight polynomial is defined, with roots $1, 2, 3, 4, 5, 6, 7, 8$. The isolator polynomials a_3 (degree 4) and b_3 (degree 3) are shown to have roots

$p(x)$:	1		2		3		4		5		6		7		8
a_3 **and** b_3**:**		1.46		2.34		3.80		4.5		5.21		6.66		7.54	

The isolator polynomials a_4 (degree 3) and b_4 (degree 4) are shown to have roots

$p(x)$:	1		2		3		4		5		6		7		8
a_4 **and** b_4**:**		1.17		2.89		3.21		4.5		5.79		6.15		7.83	

```
(* Define Mathematica functions a and b *)

a[n_Integer,f_] := PolynomialRemainder[a[n-2,f],a[n-1,f],t]
a[0,f] := D[f,t]
a[-1,f] := f
a[-2,f] := 0
q[n_Integer,f_] := PolynomialQuotient[a[n-2,f],a[n-1,f],t]
b[n_Integer,f_] := b[n-2,f] - q[n,f] b[n-1,f]
b[-1,f_] := 0
b[0,f_] := -1

(* Create sample polynomial with 8 evenly spaced roots *)

f = Expand[(t-1)(t-2)(t-3)(t-4)(t-5)(t-6)(t-7)(t-8)]
```

$$40320 - 109584t + 118124t^2 - 67284t^3 + 22449t^4 - 4536t^5 + 546t^6 - 36t^7 + t^8$$

```
(* Compute a[3,f] and b[3,f] and solve their roots *)
a3 = Expand[a[3,f]]
```

$$\frac{22851}{4} - \frac{47223}{8}t + \frac{34551}{16}t^2 - \frac{2673}{8}t^3 + \frac{297}{16}t^4$$

```
N[Solve[a3==0,t],4]
```

$$\{\{t{-}{>}\ 6.661\}, \{t{-}{>}\ 2.339\}, \{t{-}{>}\ 5.205\}, \{t{-}{>}\ 3.795\}\}$$

```
b3 = Expand[b[3,f]]
```

$$\frac{99}{64} - \frac{103}{64}t + \frac{27}{64}t^2 - \frac{t^3}{32}$$

```
N[Solve[b3==0,t],4]
```

$$\{\{t{-}{>}\ 4.5\}, \{t{-}{>}\ 7.541\}, \{t{-}{>}\ 1.459\}\}$$

```
(* Compute a[4,f] and b[4,f] and solve their roots *)
a4 = Expand[a[4,f]]
```

$$-(\frac{592896}{77}) + \frac{1307648}{231}t - \frac{9216}{7}t^2 + \frac{2048}{21}t^3$$

```
N[Solve[a4==0,t],4]
```

$$\{\{t{-}{>}\ 4.5\}, \{t{-}{>}\ 6.145\}, \{t{-}{>}\ 2.855\}\}$$

```
b4 = Expand[b[4,f]]
```

$$-(\frac{64}{7}) + \frac{3104}{231}t - \frac{12176}{2079}t^2 + \frac{32}{33}t^3 - \frac{16}{297}t^4$$

```
N[Solve[b4==0,t],4]
```

$$\{\{t{-}{>}\ 7.834\}, \{t{-}{>}\ 1.166\}, \{t{-}{>}\ 5.792\}, \{t{-}{>}\ 3.208\}\}$$

Figure 1: *Mathematica* program for computing isolator polynomials.

5 Conclusion

This paper has presented a method through which real roots of a polynomial can be isolated. A practical application of this method is to isolate real roots of polynomials for real root finding. For this application, the method is efficient for low-degree polynomials for which the degrees of the isolator polynomials are less than five.

References

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