

Short Communication

Rational hodographs

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Abstract. An equation for the hodograph control points for a rational degree n Bézier curve is derived. If, for a rational Bézier curve with positive weights, all weights are set to one, then the hodograph for this integral Bézier curve will bound the tangent directions of the rational curve.

The hodograph of a curve is defined as its first derivative [Bézier '86]. The hodograph $\mathbf{H}(t)$ of a Bézier curve $\mathbf{P}(t) = \sum_{i=0}^n \mathbf{P}_i B_i^n(t)$ is itself a Bézier curve of degree one less than the original curve:

$$\mathbf{H}(t) = \mathbf{P}'(t) = \sum_{i=0}^{n-1} \mathbf{H}_i B_i^{n-1}(t)$$

where the hodograph control points are $\mathbf{H}_i = n(\mathbf{P}_{i+1} - \mathbf{P}_i)$. Hodographs have several uses. For example, the tangent vector of a curve at $\mathbf{P}(t)$ is $\mathbf{H}(t)$. Also, hodographs indicate whether a curve segment contains a cusp. If the hodograph passes through its coordinate origin, the original curve has a cusp. Another use for hodographs is to provide a condition for assuring that two curves intersect in at most one point, or that the intersection curve of two surfaces contains no closed loops [Sederberg et al. '87].

The hodograph $\mathbf{R}(t)$ of a rational Bézier curve

$$\mathbf{P}(t) = \frac{\sum_{i=0}^n w_i \mathbf{P}_i B_i^n(t)}{\sum_{i=0}^n w_i B_i^n(t)} = \frac{\mathbf{N}(t)}{D(t)}$$

can be found using the quotient rule for derivatives:

$$\mathbf{R}(t) = \mathbf{P}'(t) = \frac{D(t)\mathbf{N}'(t) - D'(t)\mathbf{N}(t)}{D^2(t)}$$

This produces a vector valued numerator of degree $2n - 2$ and a scalar valued denominator of degree $2n$. We will refer to $D^2(t)\mathbf{R}(t) = D(t)\mathbf{N}'(t) - D'(t)\mathbf{N}(t)$ as the *scaled hodograph*. Note that the scaled hodograph can be interpreted as a degree $2n - 2$ Bézier curve $\sum_{k=0}^{2n-2} \mathbf{R}_k B_k^{2n-2}(t)$

The control points \mathbf{R}_k of the scaled hodograph are given by

$$\mathbf{R}_k = \frac{\sum_{i=\max(0, k-n+1)}^{\lfloor k/2 \rfloor} (k-2i+1) \binom{n}{i} \binom{n}{k-i+1} \mathbf{D}_{i, k-i+1}}{\binom{2n-2}{k}} \tag{1}$$

where $\mathbf{D}_{ij} = w_i w_j (\mathbf{P}_j - \mathbf{P}_i)$.

There is a simple geometric pattern in equation (1) which may not be evident at first glance. This is illuminated by examining a few cases. For a rational quadratic Bézier curve, its scaled hodograph is degree two with control points:

$$\mathbf{R}_0 = 2\mathbf{D}_{01}, \quad \mathbf{R}_1 = \mathbf{D}_{02}, \quad \mathbf{R}_2 = 2\mathbf{D}_{12}.$$

For a rational cubic Bézier curve, the scaled hodograph is degree four with control points:

$$\mathbf{R}_0 = 3\mathbf{D}_{01}, \quad \mathbf{R}_1 = \frac{3}{2}\mathbf{D}_{02}, \quad \mathbf{R}_2 = \frac{1}{2}\mathbf{D}_{03} + \frac{3}{2}\mathbf{D}_{12}, \quad \mathbf{R}_3 = \frac{3}{2}\mathbf{D}_{13}, \quad \mathbf{R}_4 = 3\mathbf{D}_{23}.$$

Consider the integral Bézier curve obtained by setting to one all the weights of a given rational Bézier curve. The control points of the scaled hodograph of the rational curve can each be expressed as weighted linear combinations of the control points of the integral hodograph \mathbf{H}_j . This is so because

$$\mathbf{D}_{i, k-i+1} = w_i w_{k-i+1} \frac{1}{n} \sum_{j=i}^{k-i} \mathbf{H}_j.$$

The authors are indebted to E.T.Y. Lee for the following derivation of equation (1) which is more direct than our original proof:

$$\begin{aligned} D^2(t)\mathbf{R}(t) &= \sum_{i=0}^n w_i B_i^n \sum_{j=0}^n w_j \mathbf{P}_j (B_j^n)' - \sum_{i=0}^n w_i \mathbf{P}_i B_i^n \sum_{j=0}^n w_j (B_j^n)' \\ &= \sum_{0 \leq i, j \leq n} w_i w_j (\mathbf{P}_j - \mathbf{P}_i) B_i^n (B_j^n)' \\ &= \sum_{0 \leq i < j \leq n} \mathbf{D}_{ij} \Psi_{ij}^n \end{aligned} \tag{2}$$

where Ψ_{ij}^n is the Wronskian,

$$\Psi_{ij}^n = B_i^n (B_j^n)' - B_j^n (B_i^n)'$$

From the derivative formula $(B_i^n)' = n(B_{i-1}^{n-1} - B_i^{n-1})$, the Wronskian can be written:

$$\Psi_{ij}^n = A - B$$

with

$$A = n(B_i^n B_{j-1}^{n-1} - B_j^n B_{i-1}^{n-1}), \quad B = n(B_i^n B_j^{n-1} - B_j^n B_i^{n-1}).$$

Applying the product formula

$$B_i^n B_j^{n-1} = \frac{\binom{i+j}{i} \binom{2n-1-i-j}{n-i}}{\binom{2n-1}{n}} B_{i+j}^{2n-1},$$

yields

$$\begin{aligned}
 A &= \frac{n}{\binom{2n-1}{n}} \binom{2n-i-j}{n-i} \left(\binom{i+j-1}{i} - \binom{i+j-1}{j} \right) B_{i+j-1}^{2n-1} \\
 &= \frac{n}{\binom{2n-1}{n}} \binom{2n-i-j}{n-i} \binom{i+j}{i} \frac{j-i}{j+i} B_{i+j-1}^{2n-1}, \\
 B &= \frac{n}{\binom{2n-1}{n}} \binom{i+j}{i} \left(\binom{2n-1-i-j}{n-i} - \binom{2n-1-i-j}{n-j} \right) B_{i+j}^{2n-1} \\
 &= \frac{n}{\binom{2n-1}{n}} \binom{2n-i-j}{n-i} \binom{i+j}{i} \frac{i-j}{2n-i-j} B_{i+j}^{2n-1}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \Psi_{ij}^n &= \frac{n}{\binom{2n-1}{n}} \binom{2n-i-j}{n-i} \binom{i+j}{i} \frac{j-i}{(i+j)(2n-i-j)} \\
 &\quad \times \left((2n-i-j) B_{i+j-1}^{2n-1} + (i+j) B_{i+j}^{2n-1} \right).
 \end{aligned}$$

Applying the degree raising formula

$$B_k^m = \frac{m+1-k}{m+1} B_k^{m+1} + \frac{k+1}{m+1} B_{k+1}^{m+1}$$

yields

$$\begin{aligned}
 \Psi_{ij}^n &= \frac{n}{\binom{2n-1}{n}} \binom{2n-i-j}{n-i} \binom{i+j}{i} \frac{j-i}{(i+j)(2n-i-j)} (2n-1) B_{i+j-1}^{2n-2} \\
 &= (j-i) \binom{n}{i} \binom{n}{j} \frac{B_{i+j-1}^{2n-2}}{\binom{2n-2}{i+j-1}}. \tag{3}
 \end{aligned}$$

Next, substitute equation (3) into equation (2), setting $k = i + j - 1$. For $0 \leq k \leq n - 1$, i runs from 0 to $\lfloor k/2 \rfloor$, and for $n \leq k \leq 2n - 2$, i runs from $k - n + 1$ to $\lfloor k/2 \rfloor$, that is,

$$\sum_{0 \leq i < j \leq n} = \sum_{k=0}^{2n-2} \sum_{i=\max(0, k-n+1)}^{\lfloor k/2 \rfloor}$$

Thus,

$$D^2(t) \mathbf{R}(t) = \sum_{k=0}^{2n-2} B_k^{2n-2} \sum_{i=\max(0, k-n+1)}^{\lfloor k/2 \rfloor} \frac{(k-2i+1) \binom{n}{i} \binom{n}{k-i+1} \mathbf{D}_{i, k-i+1}}{\binom{2n-2}{k}}.$$

References

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