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Implicitizing rational surfaces with base points using the method of moving surfaces

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Abstract. The method of moving planes and moving quadrics can express the implicit equation of a parametric surface as the determinant of a matrix M. The rows of M correspond to moving planes or moving quadrics that follow the parametric surface. Previous papers on the method of moving surfaces have shown that a simple base point has the effect of converting one moving quadric to a moving plane. A much more general version of the method of moving surfaces is presented in this paper that is capable of dealing with multiple base points. For example, a double base point has the effect (in this new version) of converting two moving quadrics into moving planes, eliminating one additional moving quadric, and eliminating a column of the matrix (i.e., a blending function of the moving surfaces)—thereby dropping the degree of the implicit equation by four. Furthermore, this is a unifying approach whereby tensor product surfaces, pure degree surfaces, and "corner-cut" surfaces, can all be implicitized under the same framework and do not need to be treated as distinct cases. The central idea in this approach is that if a surface has a base point of multiplicity k, the moving surface blending functions must have the same base point, but of multiplicity $k-1$. Thus, we draw moving surface blending functions from the derivative ideal I', where I is the ideal of the parametric equations. We explain the general outline of the method and show how it works in some specific cases. The paper concludes with a discussion of the method from the point of view of commutative algebra.

To Bruno Buchberger in honor of his achievements in computational algebra

1. Introduction

The method of moving surfaces was introduced in [**5**] as a procedure for computing the implicit equation of a rational surface. That paper presented empirical evidence that surfaces with base points can be implicitized using the method of moving surfaces, but no proof was given and no discussion was made of how to identify an appropriate set of moving surfaces with which to implicitize a surface with several base points. Subsequently, [**2**] proved that the method of moving surfaces always works if there are no base points, and [**1**] showed that it works if the base points form a local complete intersection. This paper presents a method for

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finding a collection of moving surfaces that can implicitize a surface that has a more complicated collections of base points.

Our discussion focuses on tensor-product surfaces, although we will show in Section 3 that "pure degree" surfaces can be dealt with by expressing them as tensor-product surfaces with a base point of high multiplicity at infinity. Denote $\mathbf{s} = (s, \bar{s}; t, \bar{t})$, and $\mathbf{x} = (x, y, z, w)$. A rational tensor-product surface of degree $m \times n$ in bihomogeneous form is given by the equation:

$$\mathbf{x}(\mathbf{s}) = (x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}), w(\mathbf{s})) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{x}_{ij} s^i \bar{s}^{(m-i)} t^j \bar{t}^{(n-j)} \tag{1}$$

where $x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}), w(\mathbf{s})$ are bihomogeneous of degree $m \times n$ in $s, \bar{s}, t, \bar{t}$. In the affine piece of $\mathbb{P}^3$ defined by setting $w = 1$, the coordinates of points on the surface are given by

$$x = \frac{x(\mathbf{s})}{w(\mathbf{s})}, \qquad y = \frac{y(\mathbf{s})}{w(\mathbf{s})}, \qquad z = \frac{z(\mathbf{s})}{w(\mathbf{s})}.$$

A base point is a value $\mathbf{s}$ for which $\mathbf{x}(\mathbf{s}) = (0, 0, 0, 0)$. We assume that the number of base points is finite, which is equivalent to requiring that $x(\mathbf{s})$, $y(\mathbf{s})$, $y(\mathbf{s})$, and $w(\mathbf{s})$ have no common factor. This is not a limitation, since a common factor could simply be divided out.

A *moving surface* is defined as

$$g(\mathbf{x}, \mathbf{s}) = \sum_{i=1}^{b} h_i(\mathbf{x}) \gamma_i(\mathbf{s}) = 0$$

where h_i and γ_i are polynomials. When h_i is linear or quadratic the moving surface is also called a moving plane or a moving quadric respectively. We refer to the polynomials $\gamma_i(\mathbf{s})$, $i = 1, \ldots, b$ as the *blending functions* of the moving surface, which are required to be linearly independent. A moving surface is said to *follow* the rational surface $\mathbf{x}(\mathbf{s})$ given in (1) if

$$g(\mathbf{x}(\mathbf{s}), \mathbf{s}) \equiv 0.$$

Implicitization using the method of moving surfaces hinges on the following theorem from [**5**].

THEOREM 1. *Given a set of b moving surfaces*

$$g_j(\mathbf{x}, \mathbf{s}) = \sum_{i=1}^{b} h_{ji}(\mathbf{x}) \gamma_i(\mathbf{s}) = 0, \qquad j = 1, \ldots, b,$$

each of which follows a given rational surface $\mathbf{x}(\mathbf{s})$ defined by (1). Define

$$f(\mathbf{x}) = \begin{vmatrix} h_{11}(\mathbf{x}) & \dots & h_{1b}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ h_{b1}(\mathbf{x}) & \dots & h_{bb}(\mathbf{x}) \end{vmatrix}. \tag{2}$$

If $f(\mathbf{x})$ is not identically zero (i.e., if the moving surfaces are polynomially independent) and has degree equal to the degree of the implicit equation of the rational surface $\mathbf{x}(\mathbf{s})$, then $f(\mathbf{x}) = 0$ is the implicit equation of $\mathbf{x}(\mathbf{s})$.

PROOF. See [**5**]. $\square$

The key to the method of moving surfaces is finding a square matrix (2), which means that the number of blending functions is equal to the number of linearly independent moving surfaces. Whereas previous papers on the method of moving surfaces use monomial blending functions, this paper invokes blending functions that are *poly*nomials in $K[\mathbf{s}]$. In the following discussions, we use the non-homogeneous form (that is, $\bar{s}=1$, $\bar{t}=1$) as the bihomogeneous form is needed only when the base point is at infinity.

2. A More General Method of Moving Surfaces

This section presents a method for constructing the moving surfaces required in Theorem 1 when the parametric equation (1) has base points. A heuristic is presented to account for the degree and numbers of blending functions, moving planes, and moving quadrics. Two types of base points are identified to which the heuristic applies.

2.1. What are the Blending Functions? The method of moving surfaces requires p moving planes and q moving quadrics that are polynomially independent when there are $p+q=b$ blending functions, and when $p+2q=d$ is the implicit degree of the surface being implicitized. This is a necessary and sufficient condition for the method to work. A method for finding p moving planes and q moving quadrics, for surfaces that have base points of the type described in Section 2.3, will now be given. The method usually ensures that $p+q=b>0$ and $p+2q=d>0$ but may not ensure the moving surfaces are polynomially independent. However, in every case we have examined thus far, polynomially independent moving planes and quadrics have always been found.

The key to the method of moving surfaces is finding appropriate blending functions. The main idea in our new approach is that if a surface has a base point of multiplicity k, then the moving surface blending functions must have the same base point, but of multiplicity $k-1$. Let

$$I = \langle x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}), w(\mathbf{s})\rangle$$

be the ideal generated by the polynomials in (1). Our moving planes and moving quadrics will use blending functions belonging to the derivative ideal

$$I' = \langle x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}), w(\mathbf{s}), x_s(\mathbf{s}), y_s(\mathbf{s}), z_s(\mathbf{s}), w_s(\mathbf{s}), x_t(\mathbf{s}), y_t(\mathbf{s}), z_t(\mathbf{s}), w_t(\mathbf{s})\rangle.$$

(A bihomogeneous I' will be discussed in Section 4.) What remains now is how to choose the degrees $m' \times n'$ for the blending functions. Once $m' \times n'$ have been decided, blending functions of these degrees can be easily computed from a Gröbner basis of I' (this is efficient since only two variables are involved).

2.2. Degrees of the Blending Functions. Consider the vector spaces

$$I'_{m',n'} \subset I', \quad II'_{m+m',n+n'} \subset II', \quad I^2 I'_{2m+m',2n+n'} \subset I^2 I'$$

of polynomials of degree at most $m' \times n'$, $(m+m') \times (n+n')$, $(2m+m') \times (2n+n')$ respectively. The reason for considering $II'_{m+m',n+n'}$ and $I^2 I'_{2m+m',2n+n'}$ is the maps

$$MP : {I'_{m',n'}}^4 \longrightarrow II'_{m+m',n+n'} \tag{3}$$

$$MQ : {I'_{m',n'}}^{10} \longrightarrow I^2 I'_{2m+m',2n+n'} \tag{4}$$

given by $x(\mathbf{s}), \ldots, w(\mathbf{s})$ and $x(\mathbf{s})^2, x(\mathbf{s})y(\mathbf{s}), \ldots, w(\mathbf{s})^2$ respectively. The kernel of MP gives moving planes of degree $m' \times n'$ that follow the parametrization and use blending functions from $I'_{m',n'}$. Similarly, the kernel of MQ gives the moving quadrics we want.

For the moment let us assume that a base point BP_i reduces

1. the implicit degree by d_i from $2mn$,
2. the dimension of $I'_{m',n'}$ by b_i from $(m'+1)(n'+1)$,
3. the dimension of $II'_{m+m',n+n'}$ by p_i from $(m+m'+1)(n+n'+1)$, and
4. the dimension of $I^2I'_{2m+m',2n+n'}$ by q_i from $(2m+m'+1)(2n+n'+1)$.

We call d_i, b_i, p_i, q_i the *deficiency values* of BP_i. In this section, we will assume that the deficiency effects of one base point are independent from that of other base points. (In Section 4, we will define the deficiency values rigorously and discuss what it means for the deficiency effects to be independent.)

With these assumptions, the implicit degree is

$$d = 2mn - \textstyle\sum_i d_i, \tag{5}$$

the dimension of $I'_{m',n'}$ is

$$b = (m'+1)(n'+1) - \textstyle\sum_i b_i, \tag{6}$$

the dimension of $II'_{m+m',n+n'}$ is

$$p' = (m+m'+1)(n+n'+1) - \textstyle\sum_i p_i, \tag{7}$$

the dimension of $I^2I'_{2m+m',2n+n'}$ is

$$q' = (2m+m'+1)(2n+n'+1) - \textstyle\sum_i q_i. \tag{8}$$

¿From (3), it follows easily that the number of independent moving planes is

$$p = 4b - p'.$$

A moving plane generates 4 moving quadrics. If the $4p$ moving quadrics generated by the p moving planes are linearly independent, then by (4), the number of linearly independent moving quadrics not coming from moving planes is

$$q = (10b - q') - 4p.$$

To have a square matrix of the right degree, we need $b = p+q$ and $d = p+2q$. Using the values of d, b, p, q stated above, it is easy to prove that

$$b = p + q \quad \text{and} \quad d = p + 2q \tag{9}$$

if and only if

$$\begin{aligned} (m'+1-m)(n'+1-n) &= \textstyle\sum_i (3b_i - 3p_i + q_i) \\ 3(m'+1-m)(n'+1-n) &= \textstyle\sum_i (d_i + 8b_i - 7p_i + 2q_i). \end{aligned} \tag{10}$$

One of the key ideas of our method of moving surfaces is to find collections of base points for which

$$3b_i - 3p_i + q_i = 0, \quad \text{for all } i \tag{11}$$

$$d_i + 8b_i - 7p_i + 2q_i = 0, \quad \text{for all } i. \tag{12}$$

We will define these base points in Section 2.3.

If (11) and (12) are satisfied, then (10) shows that (9) is equivalent to

$$m' = m - 1 \quad \text{or} \quad n' = n - 1. \tag{13}$$

Thus, when we use our method, one of m' or n' will be determined by (13).

However, since our $b \times b$ matrix comes from p moving planes and q moving quadrics, we also clearly need

$$b > 0, \quad p \geq 0, \quad q \geq 0. \tag{14}$$

When combined with (13) and the above formulas for d, b, p, q, a straightforward argument leads to the following inequalities.

Lemma 1. *Let d, b, p, q be as above. If $m' = m-1$, then (14) hold for blending functions from $I'_{m-1,n'}$ if and only if*

$$\max\left(\frac{\sum_i b_i + 1}{m} - 1, n - 1 + \frac{\sum_i (4b_i - p_i)}{2m}\right) \leq n' \leq 2n - 1 + \frac{\sum_i (6b_i - 4p_i + q_i)}{m},$$

and if $n' = n-1$, then (14) hold for blending functions from $I'_{m',n-1}$ if and only if

$$\max\left(\frac{\sum_i b_i + 1}{n} - 1, m - 1 + \frac{\sum_i (4b_i - p_i)}{2n}\right) \leq m' \leq 2m - 1 + \frac{\sum_i (6b_i - 4p_i + q_i)}{n}.$$

Hence, if our base points satisfy (11) and (12), then picking m', n' as described in Lemma 1 leads to a non-vacuous $b \times b$ matrix built from p moving planes and q moving quadrics whose determinant should be the implicit equation of the surface. Empirically, we have found that the method works well when the value of (m', n') is one of the values

$$(m', n') = (m-1, n-1),\ (m-1, n-2),\ (m-2, n-1).$$

2.3. Triangular and Rectangular Base Points. We have identified two special types of base points whose deficiency values satisfy equations (11) and (12).

Definition 1. *The point BP_i is a **triangular k-ple point** if there is an affine transformation sending BP_i to the origin such that the monomials of the transformed parametric polynomials do not include $s^i t^j$, $i + j < k$; furthermore, the coefficient matrix of the transformed parametric polynomials with respect to the degree k monomials has rank $k+1$.*

Definition 2. *The point BP_i is a **rectangular $k \times l$-ple point** if there is an affine transformation sending BP_i to the origin such that the monomials of the transformed parametric polynomials do not include $s^i t^j$, $i < k, j < l$; furthermore, if the generic linear combination of the transformed parametric polynomials is written as $ax(\mathbf{s}) + by(\mathbf{s}) + cz(\mathbf{s}) + dw(\mathbf{s}) = s^k P + t^l Q$, the coefficient matrix of P and Q has rank 2.*

These definitions specify that there is a triangular or a rectangular base point if the monomial support of an affine transformation of the parametric polynomials has a missing triangular or rectangular corner at the origin, provided that, for a k-ple point, all monomials of degree k must exist independently, and for a $k \times l$-ple point, the monomials s^k and t^l must exist independently. Since there are four parametric polynomials, a k-ple point must satisfy $k + 1 \leq 4$, so that $k \leq 3$.

In Section 4, we will prove that the deficiency values of k-ple and $k \times l$-ple base points are given by the following table:

BP_i	k-ple	$k \times l$-ple
d_i	k^2	kl
b_i	$\binom{k}{2}$	$(k-1)(l-1)$
p_i	$\binom{2k}{2}$	$2(k-1)(l-1)+kl$
q_i	$\binom{3k}{2}$	$3(k-1)(l-1)+3kl$

Note that the deficiency values of a k-ple point and a $k \times k$-ple point are the same when $k = 1, 2$.

The numbers in the table are easily seen to satisfy equations (11) and (12). It follows that our method applies whenever m', n' satisfy Lemma 1. Of course, this assumes that equations (5), (6), (7) and (8) hold for these values of m', n'. In Section 4, we will show that (5) is always true and that (6), (7) and (8) are true for sufficiently large m', n'. However, empirical evidence suggests that these equations also hold for those m', n' which satisfy Lemma 1.

The deficiency values in the above table can also be derived using the following heuristic. For a k-ple point, consider the sets of monomials $A = \{s^i t^j \mid k \le i+j, i \le m, j \le n\}$ and $A' = \{s^i t^j \mid k-1 \le i+j, i \le m', j \le n'\}$. Twice the area of the Newton polygon of A is $2mn - k^2$. The number of missing monomials around the origin for the sets of monomials A', AA', A^2A' are $\binom{k}{2}$, $\binom{2k}{2}$, $\binom{3k}{2}$ respectively. Similarly, for a $k\times l$-ple point, consider the sets of monomials $A = \{s^i t^j \mid k \le i \text{ or } l \le j, i \le m, j \le n\}$ and $A' = \{s^i t^j \mid k-1 \le i \text{ or } l-1 \le j, i \le m', j \le n'\}$. Twice the area of the Newton polygon of A is $2mn - kl$. The number of missing monomials around the origin for the sets of monomials A', AA', A^2A' are $(k-1)(l-1)$, $2(k-1)(l-1)+kl$, $3(k-1)(l-1)+3kl$ respectively.

2.4. Discussion. In [**1**], the method of moving surfaces dealt with a local complete intersection base point by converting one moving quadric to a moving plane, but that method is not capable of handling k-ple base points for $k \ge 2$. It is interesting to examine how the method outlined in this section handles k-ple base points. A 2-ple base point drops the degree of the implicit equation by four. But, it does not work to convert four moving quadrics to moving planes. Instead, two moving quadrics become moving planes, one moving quadric is eliminated, and one blending function is eliminated, thereby maintaining a square matrix. This usually requires the use of polynomial blending functions.

A 3-ple base point drops the degree of the implicit equation by nine. The method deals with 3-ple base points by converting three moving quadrics to moving planes, removing three moving quadrics, and removing three blending functions.

These observations are consistent with the deficiencies of a k-ple point: the number of blending functions is reduced by $b_i = \binom{k}{2}$, the number of moving planes is increased by $p_i - 4b_i = k$, and the number of moving quadrics is reduced by $4p_i - q_i - 6b_i = \binom{k+1}{2}$.

The examples presented in Section 3 indicate that the moving surface method works for base points that are more complicated than triangular and rectangular. However, we do not have a precise characterization of all base points for which we expect the method to work.

We did not prove that the determinant obtained by the moving surface method does not vanish, though this seem to be so empirically. Furthermore, the heuristic derivation for the quantities computed in equations (6), (7) and (8) assumes that the linear conditions involved are independent. This is related to the concept of regularity discussed in Section 4.

3. Examples

This section illustrates the technique with several examples. The computations are carried out in bihomogeneous form in order to properly account for base points at infinity. We use the table in Section 2.3 to count the deficiency values of the base points and Lemma 1 to predict the degrees of the blending functions.

Section 3.1 shows a surface whose parametric equations are pure degree three. In our approach, however, the surface is treated as a bicubic patch with one 3-ple base point at infinity. This example is the one used in [**5**], but in that paper, different blending functions and counting formulae were needed because pure degree surfaces behave differently from tensor-product surfaces. Using the new, unifying method, the same results are obtained without resorting to special case blending functions. Section 3.2 shows a bicubic patch with several base points of different multiplicity.

Section 3.3 gives a further example of the generality of the method by presenting a "corner-cutting" example. Section 3.4 presents an example involving a complicated base point that behaves like a combination of a 2-ple point and two 1-ple points. This paper does not discuss base points of such complexity, but the method works at least in this example. Section 3.5 gives another example involving complicated base points.

3.1. A pure degree three surface with six base points.

Consider the degree three parametric surface given by $(x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}), w(\mathbf{s})) =$

$$\begin{aligned}
&[(2,0,0,1)\bar{s}^3 + (0,0,0,0)s\bar{s}^2 + (0,0,0,0)s^2\bar{s} + (0,0,0,0)s^3]t^3\\
&+[(4,0,2,1)\bar{s}^3 + (0,-2,-3,0)s\bar{s}^2 + (0,0,0,0)s^2\bar{s} + (0,0,0,0)s^3]\bar{t}t^2\\
&+[(2,-2,-2,-1)\bar{s}^3 + (4,-1,-3,0)s\bar{s}^2 + (1,0,-2,1)s^2\bar{s} + (0,0,0,0)s^3]\bar{t}^2 t\\
&+[(2,2,0,-1)\bar{s}^3 + (3,1,-2,-1)s\bar{s}^2 + (1,-2,-3,1)s^2\bar{s} + (0,-1,-1,1)s^3]\bar{t}^3
\end{aligned}$$

whose Newton polygon is shown in Figure 1.

This surface has six finite 1-ple base points, and one infinite 3-ple base point. For $(m,n) = (3,3)$ and $m' = m-1 = 2$, the formulas from Section 2 give $(m', n') = (2,1)$, and further predict $d = 3$, $b = 3$, $p = 3$ and $q = 0$. This means that the implicit equation should be a 3×3 determinant with three linear rows. In fact, a basis of $I'_{2,1}$ is found to be $\{\bar{s}^2\bar{t}, \bar{s}^2 t, \bar{s}s\bar{t}\}$. Choosing this basis to be the blending functions, we obtain three moving planes, which form a 3×3 matrix

$$\begin{pmatrix} -2w - x & -2w + z - 2y + x & -w - z \\ z & y & x \\ y - x & -2w + x & -z + y \end{pmatrix}.$$

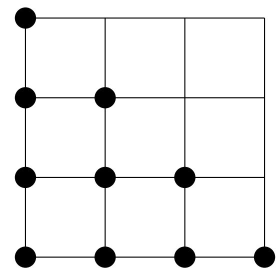

FIGURE 1. The Newton polygon for the pure degree three surface in Section 3.1.

Computing the determinant of the matrix gives the implicit equation

$$4\,w\,y\,z - w\,y^2 - 4\,x\,w^2 - 3\,x\,y^2 + z^3 - 3\,y\,z^2 + 3\,z\,y^2 +$$
$$2\,z\,w^2 - z\,x\,w - 3\,y\,x\,w + 3\,y\,x^2 + 2\,w\,x^2 - x^2\,z = 0.$$

3.2. A bicubic surface with one 3-ple, one 2-ple and two 1-ple base points. A bicubic parametric surface is defined by $(x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}), w(\mathbf{s})) =$

$$[(\frac{1}{2}, -\frac{1}{2}, 1, \frac{3}{2})\bar{s}^3 + (-1, 1, 1, -1)s\bar{s}^2 + (-\frac{1}{2}, 0, 1, 0)s^2\bar{s} + (0, 1, 1, -1)s^3]t^3 +$$
$$[(0,0,0,0)\bar{s}^3 + (-12, -\frac{21}{2}, -\frac{14}{3}, 8)s\bar{s}^2 + (-1, \frac{1}{2}, 1, 1)s^2\bar{s} + (\frac{3}{8}, -\frac{7}{2}, -\frac{119}{24}, \frac{29}{8})s^3]\bar{t}t^2$$
$$+[(0,0,0,0)\bar{s}^3 + (0,0,0,0)s\bar{s}^2 + (\frac{393}{8}, \frac{195}{8}, -\frac{83}{8}, -\frac{225}{8})s^2\bar{s} + (6, -9, -10, 6)s^3]\bar{t}^2 t$$
$$+[(0,0,0,0)\bar{s}^3 + (0,0,0,0)s\bar{s}^2 + (0,0,0,0)s^2\bar{s} + (\frac{207}{4}, \frac{45}{4}, \frac{189}{4}, -\frac{9}{4})s^3]\bar{t}^3$$

whose Newton polygon is in Figure 2.

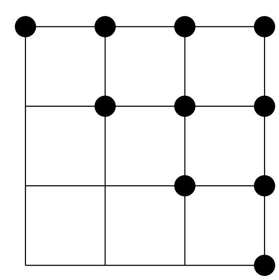

FIGURE 2. The Newton polygon for the bicubic surface in Section 3.2.

This surface has a 3-ple base point at $(0,0)$, a 2-ple base point at $(2,3)$, and two 1-ple base points at $(1,2)$ and $(-1,-2)$. Similarly, for $(m,n) = (3,3)$ and $m' = m - 1 = 2$, the formulas from Section 2 predict $(m', n') = (2,1)$ and that

$$d = 3, \quad b = 2, \quad p = 1, \quad q = 1.$$

Using Gröbner bases and $(m', n') = (2,1)$, a basis of $I'_{2,1}$ is found to be

$$\{-3s^2 + s^2 t, -\frac{3}{2}s^2 + st\}$$

With this basis we can find one moving plane and one moving quadric that give a 2×2 matrix whose determinant is the cubic implicit equation

$$\begin{aligned}&-86157\,z^3 - 333564\,w\,y\,z + 49788\,z\,x\,w + 71019\,z^2\,w + 106049\,w^3 + 459526\,w^2\,y\\&+ 509408\,y^3 - 98118\,z^2\,x + 780160\,w\,y^2 - 74934\,x\,w^2 + 1964\,w\,x^2 - 229632\,x\,y^2\\&+ 330966\,y\,z^2 - 534528\,z\,y^2 - 55551\,z\,w^2 - 9640\,y\,x^2 - 13500\,x^2\,z\\&- 245496\,y\,x\,w + 216792\,x\,z\,y + 10264\,x^3 = 0.\end{aligned}$$

3.3. A biquadratic surface with corner cutting: top-left and bottom-right. We now consider a biquadratic surface given by

$$\begin{aligned}(x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}), w(\mathbf{s})) =&[(0,0,0,0)\bar{s}^2 + (0,0,0,0)s\bar{s} + (-1,1,-2,1)s^2]t^2\\&+[(0,0,0,0)\bar{s}^2 + (0,0,0,0)s\bar{s} + (-1,-2,-2,0)s^2]\bar{t}t\\&+[(-2,-1,-2,1)\bar{s}^2 + (2,0,1,-2)s\bar{s} + (0,0,0,0)s^2]\bar{t}^2.\end{aligned}$$

Figure 3 is its Newton polygon.

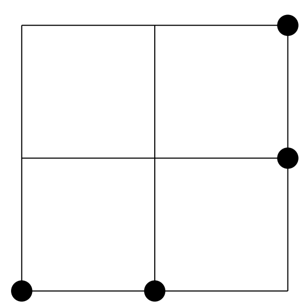

FIGURE 3. The Newton polygon for the biquadratic surface in Section 3.3.

This surface contains a 2×2-ple point at $t = \infty$, and a 1-ple point at $s = \infty$ as suggested by the parametric monomial support. If we use $(m', n') = (1, 1)$, the counting formulas predict that $d = 3, b = 3, p = 3$, and $q = 0$. Using Gröbner bases, we find a basis for $I'_{1,1}$:

$$\bar{s}\bar{t}, \quad \bar{t}s, \quad s\,t.$$

With this basis, three corresponding moving planes can be computed

$$\bar{s}\bar{t}(4\,x - 2\,y + 6\,w) + \bar{t}s\,(8\,x + 12\,y - 16\,z) + s\,t\,(27\,y - 45\,w + 18\,z - 54\,x) = 0$$

$$\bar{s}\bar{t}(8\,w + 4\,z) + \bar{t}s\,(24\,x + 36\,y - 48\,z) + s\,t\,(-170\,x + 85\,y - 147\,w + 54\,z) = 0$$

$$\bar{t}s\,(-2\,x - y - w + 2\,z) + s\;t(6\,x - 3\,y + 5\,w - 2\,z) = 0.$$

The resulting 3×3 determinant is the implicit equation

$$\begin{aligned}&784\,x^3 - 46\,y^3 - 16\,z^3 + 522\,w^3 + 2072\,w\,x^2 + 1820\,x\,w^2 + 302\,w\,y^2 - 666\,w^2\,y\\&+ 152\,z^2\,w - 604\,z\,w^2 - 968\,y\,x^2 + 380\,x\,y^2 - 124\,z\,y^2 - 72\,y\,z^2 - 688\,x^2\,z\\&+ 176\,z^2\,x - 1264\,z\,x\,w + 552\,w\,y\,z - 1640\,y\,x\,w + 592\,x\,z\,y = 0.\end{aligned}$$

3.4. A bicubic surface with complicated base points. Consider the bicubic surface where $(x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}), w(\mathbf{s}))$ is given by

$$\begin{array}{llll} [(0,0,0,0)\bar{s}^3+ & (0,0,0,0)s\bar{s}^2+(0,0,0,0)s^2\bar{s}+ & (-2,2,1,-1)s^3]t^3 \\ +[(0,0,0,0)\bar{s}^3+ & (0,0,0,0)s\bar{s}^2+(2,3,1,1)s^2\bar{s}+ & (0,0,0,0)s^3]\bar{t}t^2 \\ +[(0,0,0,0)\bar{s}^3+ & (0,2,0,1)s\bar{s}^2+(2,1,1,2)s^2\bar{s}+ & (0,0,0,0)s^3]\bar{t}^2t \\ +[(2,1,-1,1)\bar{s}^3+ & (0,0,0,0)s\bar{s}^2+(0,0,0,0)s^2\bar{s}+ & (0,0,0,0)s^3]\bar{t}^3. \end{array}$$

Its Newton polygon is shown in Figure 4.

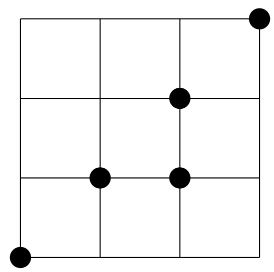

FIGURE 4. The Newton polygon for the bicubic surface in Section 3.4.

This surface has a 3-ple point at $t = \infty$. Furthermore, when $s = \infty$, there is a single complicated base point which behaves like a combination of two 1-ple points and one 2-ple point.

With $(m, n) = (3, 3)$ and $m' = m-1 = 2$, Lemma 1 suggests we use $I'_{2,1}$, where the formulas then predict that $d = 3$, $b = 2$, $p = q = 1$. Thus the implicit equation can be obtained from a 2×2 determinant with one linear row and one quadratic row. The blending functions that are chosen from $I'_{2,1}$ are $\bar{s}s\bar{t}, s^2\,t$. Using them, we get a moving plane and a moving quadric

$$s^2\,t\,(47\,x + 46\,y - 36\,w - 34\,z) + \bar{s}s\bar{t}(7\,x - 54\,y + 30\,w - 10\,z) = 0$$

$$\begin{aligned} &s^2\,t\,(10955\,x^2 - 134556\,y^2 + 36616\,y\,w - 6384\,x\,z + 54540\,w^2 - 129352\,s\,x \\ &- 81040\,x\,w + 53372\,z\,w - 1372\,z^2) - \bar{s}s\bar{t}(35532\,x\,y - 1624\,x\,z \\ &+ 159704\,y^2 + 25408\,z\,y + 99872\,s\,y\,z\,t - 49860\,w^2 + 16620\,z\,w) = 0. \end{aligned}$$

Thus computing the resulting 2×2 determinant gives the implicit equation

$$\begin{aligned} &974\,w\,x^2 + 318\,x^2\,z - 706\,y\,x^2 + 248\,y\,z^2 - 872\,z\,y^2 + 40\,z^2\,w + 416\,w\,y\,z \\ &- 624\,z\,x\,w - 40\,x\,z\,y + 1080\,w\,y^2 + 328\,y^3 - 700\,x\,y^2 + 648\,w^3 + 2152\,y\,x\,w \\ &- 313\,x^3 + 4\,z^2\,x - 56\,z^3 - 1824\,w^2\,y + 168\,z\,w^2 - 1200\,x\,w^2 = 0. \end{aligned}$$

3.5. A 5×4 surface with complicated base points. We noted in Section 2.3 that k-ple points can exist only for $k \le 3$. Yet the moving surface method also works when the parametric monomial support and the Newton polygon are in Figure 5 and in addition there is a base point at $(s, t) = (1, 2)$ which somehow behaves like a 4-ple point. The origin is like a combination of a 4-ple base point

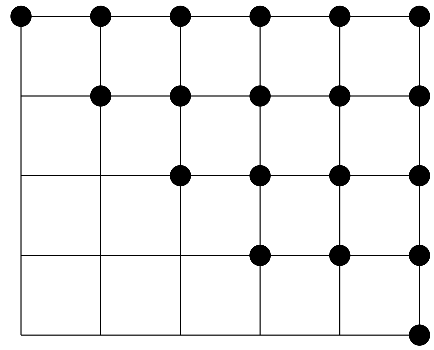

FIGURE 5. The Newton polygon for a 5×4 surface in Section 3.5.

and a 1-ple base point. This gives the following deficiency values

BP_i	4-ple	1-ple	4-ple	$\sum_i$
d_i	16	1	16	33
b_i	6	0	6	12
p_i	28	1	28	57
q_i	66	3	66	135

If we let $m' = m - 1 = 4$, then by Lemma 1, n' should satisfy $\frac{21}{10} \leq n' \leq \frac{28}{10}$. So there is no integer solution for n'! Hence we try $n' = n - 1 = 3$, which gives $m' = 3$ by Lemma 1. Now we draw the blending functions from $I'_{3,3}$, and then we have

$$\begin{aligned} b &= 4 \times 4 - 12 = 4 \\ p' &= (5+3+1)(4+3+1) - 57 = 15 \\ p &= 4 \times 4 - 15 = 1 \\ q' &= (10+3+1)(8+3+1) - 135 = 33 \\ q &= 10 \times 4 - 33 - 4 \times 1 = 3. \end{aligned}$$

The 4×4 matrix of 1 moving plane, 3 moving quadrics, and 4 blending functions indeed implicitizes the degree 7 surface.

4. Mathematical Comments

In this section, we will discuss the material presented in the previous sections from the point of view of commutative algebra. Let $R = K[\mathbf{s}] = K[s, \bar{s}; t, \bar{t}]$ be the ring of bihomogeneous polynomials over a field K of characteristic 0. We will consider the ideal

$$I = \langle x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}), w(\mathbf{s}) \rangle \subset R$$

generated by four bihomogeneous polynomials of degree $m \times n$. We will assume that the variety $\mathbf{V}(I) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is finite. A point $BP \in \mathbf{V}(I)$ is called a *base point.*

4.1. Base Points. In Section 2.3, Definitions 1 and 2 describe two types of base points, triangular k-ple and rectangular $k \times l$-ple base points. From the point of view of algebraic geometry, here are more intrinsic definitions of these base points.

Definition 3. *A base point BP of I is a **triangular k-ple point** if we can find local coordinates u and v of $\mathbb{P}^1 \times \mathbb{P}^1$ at BP such that locally near BP, the ideal I looks like*

$$\langle u^k, u^{k-1}v, u^{k-2}v^2, \ldots, u^2v^{k-2}, uv^{k-1}, v^k\rangle = \langle u, v\rangle^k \subset \mathcal{O}_{BP},$$

where $\mathcal{O}_{BP}$ is the local ring of $\mathbb{P}^1 \times \mathbb{P}^1$ at BP. In other words,

$$I_{BP} = (M_{BP})^k,$$

where $M_{BP} \subset \mathcal{O}_{BP}$ is the maximal ideal and $I_{BP} \subset \mathcal{O}_{BP}$ is generated by I.

The base points of Definition 3 have the following properties:

- A k-ple point is not a local complete intersection when $k \geq 2$ (see [**3**] for a discussion of local complete intersections).
- I has four generators, yet $\langle u, v\rangle^k$ has $k+1$ minimal generators. Thus I can have k-ple base points only for $k = 1, 2$ and 3.
- In the literature, k-ple base points are sometimes called *fat points*.

Definition 4. *A base point BP of I is a **rectangular $k \times l$-ple point** if we can find local coordinates u and v of $\mathbb{P}^1 \times \mathbb{P}^1$ at BP such that locally near BP, the ideal I looks like*

$$\langle u^k, v^l\rangle \subset \mathcal{O}_{BP}.$$

Note that the base points of Definition 4 are local complete intersections, though as we will see, they are especially nice local complete intersections.

The idea behind Definitions 1 and 2 in Section 2.3 is that local rings are hard to work with from a computational point of view. If we are given local affine coordinates for a base point, how do we determine if it satisfies Definition 3 or 4 without using local rings? This is exactly what Definitions 1 and 2 do: one can prove without difficulty that any base point satisfying Definition 1 (resp. Defintion 2) is a k-ple base point (resp. $k \times l$-ple base point) in the sense of Definition 3 (resp. Definition 4).

4.2. The Derivative Ideal and its Saturation. Given our ideal $I \subset R = K[\mathbf{s}] = K[s, \bar{s}; t, \bar{t}]$, we can form the derivative ideal of partial derivatives

$$\mathrm{D}(I) = \langle f(\mathbf{s})_s, f(\mathbf{s})_{\bar{s}}, f(\mathbf{s})_t, f(\mathbf{s})_{\bar{t}} : f(\mathbf{s}) \in I\rangle$$

where "D" stands for "derivative". Since $I = \langle x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}), w(\mathbf{s})\rangle$, we have

$$\begin{aligned}\mathrm{D}(I) = \langle & x(\mathbf{s})_s, x(\mathbf{s})_{\bar{s}}, x(\mathbf{s})_t, x(\mathbf{s})_{\bar{t}}, y(\mathbf{s})_s, y(\mathbf{s})_{\bar{s}}, y(\mathbf{s})_t, y(\mathbf{s})_{\bar{t}},\\ & z(\mathbf{s})_s, z(\mathbf{s})_{\bar{s}}, z(\mathbf{s})_t, z(\mathbf{s})_{\bar{t}}, w(\mathbf{s})_s, w(\mathbf{s})_{\bar{s}}, w(\mathbf{s})_t, w(\mathbf{s})_{\bar{t}}\rangle.\end{aligned}$$

Also observe that $I \subset \mathrm{D}(I)$ follows from the Euler formulas

$$mf(\mathbf{s}) = sf(\mathbf{s})_s + \bar{s}f(\mathbf{s})_{\bar{s}} \quad \text{and} \quad nf(\mathbf{s}) = tf(\mathbf{s})_t + \bar{t}f(\mathbf{s})_{\bar{t}} \tag{15}$$

satisfied by any bihomogeneous polynomial of degree $m \times n$. It follows in particular that $\mathbf{V}(\mathrm{D}(I)) \subset \mathbf{V}(I)$, so that every base point of $\mathrm{D}(I)$ is a base point of I.

The ideal $\mathrm{D}(I)$ is closely related to the ideal I' defined in Section 2.1. There, we worked on the affine piece of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $\bar{s} = \bar{t} = 1$, and $I' \subset K[s, t]$ was defined using $x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}), w(\mathbf{s})$ together with their partial derivatives with respect to s and t. It is easy to show that

$$\mathrm{D}(I)\big|_{\bar{s}=\bar{t}=1} = I'.$$

(First, observe that I' is contained in the left-hand side since $I \subset \mathrm{D}(I)$. Then the opposite inclusion follows by setting $\bar{s} = \bar{t} = 1$ in (15).)

However, it turns out that $\mathrm{D}(I)$ is not really the ideal we want to use as the bihomogeneous version of I'. This can be seen in Example 3.2, which uses $I'_{2,1}$. This example has $m = n = 3$, so that $\mathrm{D}(I)$ is generated by polynomials of degree 2×3 and 3×2. Thus $\mathrm{D}(I)_{2,1} = \{0\}$. It follows that the correct bihomogeneous version of the ideal I' defined in Section 2.1 is the saturation

$$I' = \mathrm{sat}(\mathrm{D}(I)). \tag{16}$$

This means that I' consists of all bihomogeneous polynomials which dehomogenize to elements coming from $\mathrm{D}(I)$ on all four affine pieces of $\mathbb{P}^1 \times \mathbb{P}^1$. Note also that I' and $\mathrm{D}(I)$ have the same base points.

We can also see that (16) is the correct ideal to use by considering the case when all the base points are 1-ple ($\Leftrightarrow$ 1×1-ple $\Leftrightarrow$ simple). Then we want I' to be the full ring R. But $\mathrm{D}(I)$ can't be the whole ring since it is generated by elements of degree $(m-1) \times n$ and $m \times (n-1)$. However, $\mathrm{D}(I)$ also has no base points, so that its saturation I' is the whole ring, as desired.

Some other nice features of I' are the following (we omit the straightforward proofs):

- If BP is a k-ple base point of I, then BP is a $(k-1)$-ple base point of I'.
- If BP is a $k \times l$-ple base point of I, then BP is a $(k-1) \times (l-1)$-ple base point of I'.

The second bullet shows that $k \times l$-ple base points are especially nice, since in general a local complete intersection base point of I need not be a local complete intersection for I'.

Later in the section we will explain why the ideal I' is needed in order to make the method of moving surfaces work.

4.3. Multiplicity and Degree. Suppose that the ideal I has base points BP_i. Then for each base point BP_i, Section 2.2 defines *deficiency values*, denoted d_i, b_i, p_i, q_i. These numbers can be defined intrinsically as follows.

First recall from [**3**] that given a base point BP of an arbitrary ideal $\tilde{I}$ of R, we have the following definitions:

- The *degree* of $\tilde{I}$ at BP is
$$\deg(\tilde{I}, BP) = \dim \mathcal{O}_{BP}/\tilde{I}_{BP}.$$
- The *multiplicity* of $\tilde{I}$ at BP is
$$e(\tilde{I}, BP) = \dim \mathcal{O}_{BP}/\langle f, g\rangle,$$
where $\langle f, g\rangle$ is the ideal of $\mathcal{O}_{BP}$ generated by generic linear combinations of the generators of $\tilde{I}$.

Now let BP_i be a base point of our ideal I, and let I' be as above. Then BP_i has the following deficiency values:

$$\begin{aligned} d_i &= e(I, BP_i) \\ b_i &= \deg(I', BP_i) \\ p_i &= \deg(II', BP_i) \\ q_i &= \deg(I^2 I', BP_i). \end{aligned}$$

Intuitively, d_i tells us how the base point affects the implicit degree, while b_i, p_i and q_i give the number of conditions imposed by the base point BP_i relative to I', II' and I^2I' respectively.

For k-ple and $k \times l$-ple base points, these values are easy to compute. For example, let BP_i be a k-ple base point. It is well-known that in a regular local ring of dimension 2, the kth power of the maximal ideal has multiplicity k^2. Furthermore, since $I_{BP_i} = \langle u, v\rangle^k$, we also have

$$\begin{aligned} I'_{BP_i} &= \langle u, v\rangle^{k-1} \\ (II')_{BP_i} &= \langle u, v\rangle^k \langle u, v\rangle^{k-1} = \langle u, v\rangle^{2k-1} \\ (I^2I')_{BP_i} &= (\langle u, v\rangle^k)^2 \langle u, v\rangle^{k-1} = \langle u, v\rangle^{3k-1}. \end{aligned}$$

This gives the numbers $b_i = \binom{k}{2}$, $p_i = \binom{2k}{2}$, $q_i = \binom{3k}{2}$ which appear in the table in Section 2.3. The numbers in this table for $k \times l$-ple base points are equally easy to compute.

4.4. Regularity and Saturation. Now assume that I has base points BP_i with deficiency numbers d_i, b_i, p_i, q_i as defined above. Our next task is to understand equations (5) to (8) in Section 2.2. Equation (5) asserts that

$$d = 2mn - \textstyle\sum_i d_i.$$

Since d_i is the multiplicity of BP_i, this is just the usual formula for the degree of the implicit equation.

The other three equations are more complicated. In fact, they may fail for some m' and n' but are true for m' and n' sufficiently large.

Let's begin with the formula for $\dim I'_{m',n'}$ given in (6). For a base point BP_i, a polynomial of degree $m' \times n'$ gives an element of the local ring $\mathcal{O}_{BP_i}$. Hence we get a map

$$\phi_{m',n'} : R_{m',n'} \longrightarrow \bigoplus_i \mathcal{O}_{BP_i}/I'_{BP_i}. \tag{17}$$

The kernel of $\phi_{m',n'}$ consists of those polynomials of degree $m' \times n'$ that satisfy the conditions imposed by the base points of I'. Since I' is saturated, the kernel of $\phi_{m',n'}$ is precisely $I'_{m',n'}$. Furthermore, as explained in [**3**], $\phi_{m',n'}$ is onto (i.e., the conditions imposed by the base points are independent) when m' and n' are sufficiently large.

It follows that (17) is onto with kernel $I'_{m',n'}$ for m', n' large. Then linear algebra implies that

$$\dim I'_{m',n'} = \dim R_{m',n'} - \textstyle\sum_i \dim \mathcal{O}_{BP_i}/I'_{BP_i} = (m'+1)(n'+1) - \sum_i b_i$$

provided that m', n' are large. This is equation (6) from Section 2.2.

Equations (7) and (8) are similar, with the twist that II' and I^2I' need not be saturated. For II', this means the following. Similar to (17), we have a map

$$\psi_{m+m',n+n'} : R_{m+m',n+n'} \longrightarrow \bigoplus_i \mathcal{O}_{BP_i}/II'_{BP_i}.$$

In this situation, the kernel of $\psi_{m+m',n+n'}$ is the degree $(m+m') \times (n+n')$ part of the saturation of II', which we write as $\mathrm{sat}(II')_{m+m',n+n'}$. Fortunately,

- $\mathrm{sat}(II')_{m+m',n+n'} = II'_{m+m',n+n'}$ for m', n' large.
- $\psi_{m+m',n+n'}$ is onto for m', n' large.

Then arguing as above gives the formula

$$\begin{aligned}\dim II'_{m+m',n+n'} &= \dim R_{m+m',n+n'} - \textstyle\sum_i \dim \mathcal{O}_{BP_i}/II'_{BP_i}\\ &= (m+m'+1)(n+n'+1) - \textstyle\sum_i p_i\end{aligned}$$

provided that m', n' are large. We also observe that

$$I_{m,n}I'_{m',n'} = II'_{m+m',n+n'}$$

since I is generated by elements of degree $m \times n$. This shows that equation (7) from Section 2.2 is true for m', n' large. The proof of equation (8) is similar.

But what does "large" mean? The answer, as explained in [**3**], involves regularity. This concept has been studied extensively for homogeneous polynomials, but the theory of regularity for bihomogeneous polynomials is just beginning (see the paper by Hoffman and Wang listed in the references to [**3**]).

Let us say that I' is (m', n')-*regular* if equation (6) is true for degree $m' \times n'$ (this isn't perfect terminology but should suffice for the following discussion). Similarly, we say that II' (resp. I^2I') is $(m+m', n+n')$-*regular* (resp. $(2m+m', 2n+n')$-*regular*) if equation (7) (resp. (8)) is true in degree $(m+m') \times (n+n')$ (resp. $(2m+m') \times (2n+n')$).

It follows that to use the method of moving surfaces in this situation, one needs to know the regularity of the ideals I', II' and I^2I'. As noted in Section 2.2, empirical evidence suggests that we should have the desired regularity when

$$(m', n') = (m-1, n-1),\ (m-1, n-2),\ (m-2, n-1). \tag{18}$$

To give a hint of what sort of mathematical theory might be involved, consider the case of no base points studied in [**2**]. Here, we know that $I' = R$, so that any (m', n') works for I'. However, one of the key points in [**2**] is the proof that

$$MP \text{ has maximal rank} \Longrightarrow MQ \text{ has maximal rank.}$$

In terms of regularity, one can show that this is equivalent to showing that

$$I \text{ is } (2m-1, 2n-1)\text{-regular} \Longrightarrow I^2 \text{ is } (3m-1, 3n-1)\text{-regular.} \tag{19}$$

In the case when base points are present, it is possible that I' is (m', n')-regular for the pairs (m', n') listed in (18). Then, in analogy with (19), it might be possible to prove that

$$II' \text{ is } (m+m', n+n')\text{-regular} \Longrightarrow I^2I' \text{ is } (2m+m', 2n+n')\text{-regular.}$$

The idea is that when the ideal II' is multiplied by I to give I^2I', the regularity should increase by the degree of the generators of I. In the case of no base points, this is what was proved in [**2**].

4.5. Why We Use I'. Our final task is to explain why I' is the correct thing to use. A first observation is that when all of the base points are local complete intersections, the results of [**1**] show that the usual method of moving surfaces works fine. So if there are no k-ple base points with $k \geq 2$, then we don't need I' (though we will say more about this below).

Furthermore, as observed in the subsection "Moving Planes and LCI Base Points" of Section 3 of [**3**], the usual method of moving surfaces is guaranteed to fail in the presence of base points that aren't local complete intersections. So we clearly need to do something different if we want the method to work with k-ple base points with $k \geq 2$.

But why is I' the proper thing to use? A first answer is that it works because the numbers add up correctly. To understand this, consider the simple case described in Section 2.4 of a single k-ple base point, where $k \geq 2$. With no base points, the method of moving planes and quadrics uses the maps

$$\begin{aligned} MP &: {R_{m-1,n-1}}^4 \longrightarrow R_{2m-1,2n-1} \\ MQ &: {R_{m-1,n-1}}^{10} \longrightarrow R_{3m-1,3n-1} \end{aligned} \tag{20}$$

given by $x(\mathbf{s}), \ldots, w(\mathbf{s})$ and $x(\mathbf{s})^2, x(\mathbf{s})y(\mathbf{s}), \ldots, w(\mathbf{s})^2$ respectively. As in [**2**], we expect MP to be an isomorphism and MQ to give mn independent moving quadrics that follow the parametrization. This gives a $mn \times mn$ matrix M with quadratic entries, so that the implicit degree is $2mn$.

Imposing a k-ple base point drops the implicit degree by k^2. Yet MP now gives $\binom{k+1}{2}$ linearly independent moving planes. If we put these in our matrix, then $\binom{k+1}{2}$ quadratic rows become linear, so that the degree drops by $\binom{k+1}{2}$. Since $k \geq 2$, this differs from k^2. Hence we have a problem.

The method of Section 2 avoids this problem by making the whole matrix smaller. In (20), we allow moving planes built out of arbitrary polynomials of degree $(m-1) \times (n-1)$. This is what enables us to use monomials as blending functions. To make the matrix smaller, we use moving planes built out of a subspace of $R_{m-1,n-1}$, so that our blending functions come from a basis of this subspace. Thus the dimension of the subspace dictates the size of the matrix.

We will use $I'_{m-1,n-1}$ as the subspace. The expected dimension (assuming regularity) is $mn - \binom{k}{2}$. Then the map MP of (20) becomes the map

$$MP : {I'_{m-1,n-1}}^4 \longrightarrow II'_{2m-1,2n-1}$$

defined in (3). Since this map is onto, the formulas of Section 2 imply that the expected dimension of the kernel is k. The corresponding moving planes give k linear rows in our matrix M, which is now of size $(mn - \binom{k}{2}) \times (mn - \binom{k}{2})$. Since the remaining rows are quadratic, the degree of M is

$$k + 2(mn - \tbinom{k}{2} - k) = 2mn - k^2.$$

This is the desired degree. Furthermore, as shown by the formulas of Section 2, we expect to find precisely $mn - \binom{k}{2} - k$ moving quadrics that follow the parametrization, don't come from moving planes, and use blending functions from $I'_{m-1,n-1}$. The method really works!

We should also note that this method is also useful in the case when only rectangular base points are present. The idea is that although we could use the method of [**1**], the method of Section 2 gives smaller matrices because $I'_{m-1,n-1}$ is strictly smaller than $R_{m-1,n-1}$ whenever there is a $k \times l$-ple point with $kl > 1$.

Although I' works, it was not easy to discover. One early example was the case of a degree n parametric curve $\mathbf{r}(t) = (x(t), y(t), z(t))$. If there are no base points, then by [**5**] we can always choose $1, t, \ldots, t^{n-1}$ as the blending functions. Otherwise, if $\mathbf{r}(t)$ has base points, then $x(t), y(t), z(t)$ have a common divisor $h(t)$ of degree $m > 0$. Then $\frac{1}{h(t)}\mathbf{r}(t)$ is a degree $n-m$ curve which can be implicitized using blending functions $1, t, \ldots, t^{n-m-1}$. The key observation is that we can regard this as using the blending functions $h(t), t\,h(t), \ldots, t^{n-m-1}h(t)$ for $\mathbf{r}(t)$. The resulting moving lines give a matrix whose determinant is the implicit equation of $\mathbf{r}(t)$. Also note that the latter blending functions are not necessarily monomials and share the same base points as the parametrization.

In the surface case, numerical experiments were tried with many different types of blending functions. For 2-ple (resp. 3-ple) base points, using blending functions which were 1-ple (resp. 2-ple) at the base points worked. This suggested that for k-ple base points, we should use blending functions which were $(k-1)$-ple at the same base points. Then the derivative ideal I' entered the picture when we realized that it automatically reduces the base points from k-ple to $(k-1)$-ple.

What's missing is an intrinsic understanding of why I' works so well. One relevant idea might be the following. For the polynomial ring $K[x]$, an element $a \in K$ gives the linear map $K[x] \to K$ defined by $P(x) \mapsto P(a)$ whose kernel consists of those polynomials vanishing at a. Similarly, we have the linear map $P(x) \mapsto P'(a)$, whose kernel contains polynomials with a double root at a. As shown in [**4**], these ideas lead to differential conditions which define arbitrary base points *and* their multiplicities. This might lead to a better understanding of why I' is the right thing to use.

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