150 points
Saturday, 20 April 2002, 3:00p—6:00p

Two pages of notes allowed.
Show all of your work.
If you think you might not understand what a certain question is asking, clearly state what you are assuming it is asking.

1. (10 points) Multiple choice. If for \( n = 10 \), the algorithm requires \( k \) arithmetic operations, approximately how many arithmetic operations would the algorithm take if \( n = 20 \)? Possible answers are:

   a. \( k \)  
   b. \( 2k \)  
   c. \( 4k \)  
   d. \( 8k \)  
   e. \( \geq 16k \)

   ____ a. The de Casteljau algorithm for a degree \( n \) Bézier curve.

   ____ b. The algorithm for converting the parametric equations of a degree \( n \) Bézier curve to power basis.

   ____ c. Given \( n + 1 \) points \( P(0), P(\delta), P(2\delta), \ldots, P(n\delta) \), on a curve known to be a degree \( n \) polynomial Bézier curve (but you don’t know the control points) find \( P((n+1)\delta) \) using forward differencing.

   ____ d. Evaluate a degree \( n \) polynomial using Horner’s method.

   ____ e. Neville’s algorithm for a degree \( n \) curve.

   ____ f. Evaluate a degree \( n \) Newton polynomial (you already know the coefficients of the polynomial).

   ____ g. Degree elevate a degree \( n \) Bézier curve.

   ____ h. Find the minimum number of line segments (with evenly spaced parameter values) needed to plot a degree \( n \) Bézier curve so that the error is less than a specified value.

   ____ i. Subdivide a degree \( n \times n \) tensor-product Bézier surface patch.

   ____ j. Compute the curvature at \( t = 1 \) for a degree \( n \) rational Bézier curve.
2. (8 points) A cubic polynomial $f(t)$ satisfies

$$f(0) = 1; \quad f(1) = 2; \quad f(3) = 10; \quad f(5) = 66.$$ 

Find $f(2)$ using Newton polynomials.

3. (8 points) A polynomial curve $P(t)$ interpolates four points as follows:

$$P(0) = (1, 2); \quad P(1) = (0, 1); \quad P(3) = (4, 2); \quad P(4) = (9, 3);$$

Use Neville’s scheme to find $P(2)$. 
4. (3 points) Using lexicographic ordering with $x > y > z$, circle the leading term of each of the following polynomials:
   a. $2x + 3y + 4z$
   b. $x^2 + xy^3 + z^4$
   c. $xy^3 + x^2z + xyz - 6x^3$

5. (3 points) Using deglex ordering with $z > y > x$, circle the leading term of each of the following polynomials:
   a. $2xy + 3yz + 4zx$
   b. $x^2z + y^3z^2 + z^4$
   c. $xy^3 + x^2z$

6. (3 points) Find the S-polynomial of $xy + x - 1$, and $x^2 - y$ using lexicographical ordering with $y > x$.

7. (10 points) Some uninformed authors suggest creating a Bézier end condition on a B-spline curve by placing several B-spline control points on top of each other, as illustrated in this figure. How many Bézier curves comprise this cubic B-spline curve? 

   What are the Cartesian coordinates for the Bézier control points for each of those curves?

   P₀ = P₁ = P₂ = (0,0) P₃ = (0.12) P₄ = (24,12)

   P₅ = (12, 0) Knot Vector = [0 1 2 3 4 5 5 5 ]

   Why is creating a Bézier end condition in this manner not as desirable as using multiple knots?
8. (3 points) Fill in the blanks. If \( \{f, g\} \) is a Gröbner basis for \( \langle f, g \rangle \) and \( p \in \langle f, g \rangle \), then the __________ of \( p \) can be evenly divided by the __________ of either \( f \) or \( g \).

9. (4 points) Given an ideal \( I = \langle x^2 + 2xz + 5z^2 - 1, xy + 3z - 2, x^3y - 5xz^2 - 2 \rangle \), if you want to find a polynomial \( f(x) \) that belongs to \( I \), you can compute the Gröbner basis using which term order? (Note that \( f(x) \) is a polynomial in \( x \) only and does not involve \( y \) or \( z \))
   a. deglex with \( x > y > z \)
   b. lexicographic with \( x > y > z \)
   c. deglex with \( z > y > x \)
   d. lexicographic with \( z > y > x \)

10. (5 points) Find the \((x, y)\) coordinates of the double point on the curve \( f(x, y) = xy - x - 2y^2 + 2y = 0 \).

11. (6 points) Two circles are given by the equations \( f(x, y) = x^2 + y^2 + 2x - y = 0 \) and \( g(x, y) = x^2 + y^2 + x - y = 0 \). Find the \((x, y)\) coordinates of the two real points at which these two circles intersect by using the Gröbner basis idea as follows.
   a. Find a polynomial in \( \langle f, g \rangle \) that only contains \( x \) and not \( y \). [Hint: computer the S-polynomial of \( f \) and \( g \).]
   b. Solve for the \( x \) coordinates of the intersection points.
   c. Now that you know the \( x \) coordinates of the intersection points, solve for the \( y \) coordinates.
12. (8 points) Use the error bounds formula for Lagrange interpolation to find an upper bound on the distance between this cubic Bézier curve and the line segment $P_0 - P_3$.

$P_0 = (0,0)$
$P_1 = (1,8)$
$P_2 = (6,11)$
$P_3 = (12,9)$

13. (6 points) For the curve in problem 12, find an upper bound on the distance from the curve to the line segment $P_0 - P_3$ using the convex hull property.

14. (8 points) How many line segments are needed to plot the curve in problem 12 if the maximum deviation between the curve and each line segment is not to exceed 0.001? (The line segments are along equally-spaced parameters)
15. (5 points) Use Euclid’s algorithm to find the GCD of 301 and 473.

16. (6 points) Find the vector that is normal to the bi-quadratic surface patch with control points:

\[ P_{02} = (0, 0, 8) \quad P_{12} = (4, 8, 8) \quad P_{22} = (8, 0, 8) \]
\[ P_{01} = (0, 0, 4) \quad P_{11} = (4, 16, 4) \quad P_{21} = (8, 0, 4) \]
\[ P_{00} = (0, 0, 0) \quad P_{10} = (4, 8, 0) \quad P_{20} = (8, 0, 0) \]

at \( s = 0 \quad t = 0 \).

17. (6 points) What are the \((x, y, z)\) coordinates of \( P(\frac{1}{2}, \frac{1}{2}) \) for the surface in problem 16?
18. (14 points) Circle T (for TRUE) or F (for FALSE) for each statement:

T    F   Any curve which obeys the convex hull property is coordinate system independent.
T    F   A curve which is variation diminishing obeys the convex hull property.
T    F   The Lagrange polynomial basis obeys the convex hull property.
T    F   If a polynomial \( f(t) \) has exactly one double root, the resultant of \( f(t) \) and \( f'(t) \) is one.
T    F   Any degree four B-spline that has a knot vector \([0,1,2,3,4,5,6,7,8]\) is at least \( C^3 \) at \( t = 4 \).
T    F   The genus of a degree \( n \) rational Bézier curve can be any number between 0 and \( (n - 1) \ast (n - 2) / 2 \), depending on the number of double points it has.
T    F   Refinement rules for Catmull-Clark subdivision surfaces are based on the idea of B-spline knot insertion.

19. (8 points) Use Bézier clipping to find values of \( t \) for which the curve \( P(t) \) does not intersect the curve \( Q(s) \).

\[
P_0 = (1,0) \quad P_1 = (1,1) \quad P_2 = (2,2) \quad P_3 = (3,3) \quad P_4 = (0,4)
\]
\[
Q_0 = (0,0) \quad Q_1 = (4,4)
\]

20. (5 points) Find an inversion equation for the curve

\[
x = \frac{t^2 - 1}{t^2 + 2t + 1}; \quad y = \frac{t^2 + 3t + 2}{t^2 + 2t + 1}.
\]

Use it to find the parameter value at the point \((1,0)\).

Equation: \( \) \( \) \( t \) at \((1,0)\): \( \)
21. (4 points) A rational quadratic Bézier curve has a curvature of $\kappa$ at the point where $t = 0$ on the curve. The curve is changed by multiplying $w_0$ by 8, $w_1$ by 4, and $w_2$ by 2. The new curvature at $t = 0$ is:

a. $2\kappa$

b. $\kappa$

c. $\kappa/2$

d. $\kappa/4$

e. Depends on the control points.

22. (5 points) Using the standard notation for interval arithmetic $[a, b] = \{x | a \leq x \leq b\}$, evaluate $[1, 2](1 - t) + [3, 4]t$ at $t = .5$ and at $t = 3$.

Answer: __________

23. (5 points) Where does point $P$ move to when this FFD is applied?

24. (3 points) A degree $n$ uniform B-spline with $m$ control points in general position has

- curve segments,
- knots in the knot vector
- order continuity at knots.

25. (4 points) Convert the polynomial $y = 10t^3$ to a degree 5 Bernstein basis polynomial.

ANSWER: $y = \underline{\_\_\_\_} B_0^5(t) + \underline{\_\_\_\_} B_1^5(t) + \underline{\_\_\_\_} B_2^5(t) + \underline{\_\_\_\_} B_3^5(t) + \underline{\_\_\_\_} B_4^5(t) + \underline{\_\_\_\_} B_5^5(t)$. 